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ABSTRACT

This is a SMSG geometry text for elementary teachers. This volume has been prepared to help elementary teachers develop a sufficient subject matter competence in the mathematics of the elementary school program. The editors feel that elementary teachers need a thorough discussion of all the materials they might teach in grades 4, 5, and 6, from a higher point of view, but presented in much the same way they would present it. The content is the same as the 7th and 8th grade SMSG course of study, but carefully edited and presented in a manner compatible with its purpose. Chapter topics include: (1) non-metric geometry; (2) measurement; (3) parallelograms and triangles; (4) constructions and congruent triangles; (5) similar triangles and variation; (6) volumes and surface areas; (7) circles and spheres; and (8) relative error. (MP)

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STUDIES IN MATHEMATICS

VOLUME VII

Intuitive Geometry

(preliminary edition)

SE 025 371

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Prepared under the supervision of an Editorial Committee consisting of:

R. D. Anderson, Louisiana State University

John A. Brown, University of Delaware

Mildred Cole, East Aurora Junior High School,
Aurora, Illinois

B. H. Colvin, Boeing Scientific Research Laboratories

H. M. Gehman, University of Buffalo

B. W. Jones, University of Colorado

H. T. Karnes, Louisiana State University

Nick Lovdjieff, Anthony Junior High School,
Minneapolis, Minnesota

J. L. Marks, San Jose State College

J. R. Mayor, American Association for the Advancement of Science

Muriel Mills, Hill Junior High School, Denver, Colorado

Veryl Schult, Washington Public Schools, Washington, D.C.

J. R. Smart, San Jose State College

Rothwell Stephens, Knox College

G. C. Webber, University of Delaware

A. B. Willcox, Amherst College

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PREFACE

Mathematics is fascinating to many persons because of its utility and because it presents opportunities to create and discover. It is continuously and rapidly growing because of intellectual curiosity, practical applications and the invention of new ideas.

The many changes which have taken place in the mathematics curriculum of the junior and senior high schools have resulted in an atmosphere in which mathematics has found acceptance in the public mind as never before. The effect of the changes has begun to influence the elementary school curriculum in mathematics as well. It is generally accepted that the present arithmetic program in grades 1-6 will be substantially changed in the next five years.

The time is past when elementary teachers can be content that they have taught mathematics if they have taught pupils to compute mechanically. More and more mathematics will be taught in the elementary grades; one important criterion of the effectiveness of a school's program will be the extent to which pupils understand. If mathematics is taught by people who do not like, and do not understand, the subject, it is highly probable that pupils will not like, and will not understand, it as well.

The School Mathematics Study Group materials for grades 4, 5, and 6 contain sound mathematics, presented at such a level that it can be understood by children. Experimental classes have found the program stimulating, and teachers have been enthusiastic about the results. Teachers in the experimental programs have received the special services of a consultant to help them with the subject matter. It is not surprising that teachers who have been trained to teach arithmetic in the traditional sense need to study more mathematics if they are to have confidence in their ability to introduce new ideas. They must know far more than the students, and understand something of the later implications of the topics they teach.

This volume, Intuitive Geometry, and a companion volume, Number Systems, have been prepared by the School Mathematics Study Group to help elementary teachers develop a sufficient subject matter competence in the mathematics of the elementary school program. What kind of material should teachers at this level have? Is a course in calculus, or abstract algebra, or applications of arithmetic what they need? The opinion of the editors is that elementary teachers need a thorough discussion of all the materials they might teach in grades 4, 5, and 6, from a higher point of view, but in a very similar setting, presented in much the same way they would present it, so that they themselves might experience something of the joy of discovery and accomplishment in mathematics which they may expect from their own pupils.

With the ideas in mind which have been expressed in the paragraph above, the existing 7th and 8th grade SMSG course of study was decided upon as the content which would be of greatest benefit to elementary teachers. The material has been carefully edited, with the idea that it must now serve a much different purpose than that for which it was originally intended. You, as the mature individual, will be able to appreciate much that would escape junior high school students. Even though, in some cases, your technical ability may not be well-developed, you will be able to think critically and to make connections between what you know and new material. In some cases, you will be surprised to find that "new" ideas in mathematics are really only a new language for ideas you have known either implicitly or explicitly for years. While your ability to compute may be improved only slightly as you study this book, you will find that you understand many operations and concepts that were previously vague or even merely tricks used to get correct answers.

In elementary mathematics today, the properties of the numbers are considered to be as important as the actual computations with numbers. The nature of the operations with numbers is considered to be as important as the answers obtained. But in addition to the work with numbers, the ideas of geometry must also be taught. Both the applications of number in geometry (measurement) and the relationships between geometric elements independent of number help to form a foundation for the later study of geometry. The introduction of basic mathematical ideas in the elementary grades is the opportunity for which you are preparing yourself.

Chapter 1

NON-METRIC GEOMETRY I

1-1. Points, Lines, Planes, and Space

Points

A geometric point is thought of as being so small that it has no size. In geometry no definition is given for the term "point". Instead many properties of points are described.

Space

You can think of space as being a set of points. There is an unlimited number of points in space. In a way, you can think of the points of space as being described or determined by position--whether they are in this room, in the world, or in the universe.

Lines

For us, a line is a set of points in space, not *any* set of points but a particular type of set of points. The term "line" means "straight line". A geometric line extends without limit in each of two directions. It does not stop at a point.

A line has certain properties relative to points and space. The first of these which we mention is:

Property 1: *Through any two different points in space there is exactly one line.*

From this one can see that there is an unlimited number of lines in space.

By using lines you can get a good idea of what space is like. Consider a point at a top corner of your desk. Now consider the set of all points suggested by the walls, the floor, and the ceiling of your classroom. Then for each point of this set there is a line through it and the selected point on your desk. Each line is a set of points. Space is made up of all the points on all such lines. Remember, these lines extend outside the room.

Just as "point" and "line" are not precisely defined, "space" is not precisely defined.

Planes

Mathematicians think of a plane as a set of points in space. It is not just any set of points, but a particular kind of set. You have already seen that a line is a set of points in space, a particular kind of set and a different kind from that of a plane. A plane is a mathematician's way of thinking about the "ideal" of a flat surface.

If two points are marked on the chalkboard, exactly one line can be drawn through these points. This is Property 1. This line is on the chalkboard. The plane represented by the chalkboard contains the set of points represented by the line which you have drawn.

Think of two points marked on a piece of plywood. Part of the line through these points can be drawn on the plywood (recall that "line" means "straight line"). Must the line through these two points be on the plane of the plywood? These are examples of:

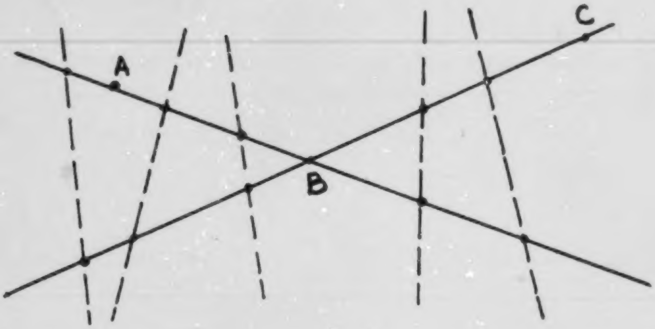
Property 2: If a line contains two different points of a plane, it lies in the plane.

If one thinks of the ends of the binding of a notebook as a pair of points, he can see that the planes represented by the pages of the notebook contain these points. The question might be asked, "How many planes contain a specified pair of points?" The notebook with its sheets spread apart suggests that there are many planes through a specified pair of points.

Suppose next that you have three points not all on the same line. Three corners of the top of a desk is an example of this. The bottoms of the legs of a three-legged stool is another example. Such a stool will stand solidly against the floor, while a four-legged chair does not always do this unless it is very well constructed.

Property 3: Any three points not on the same line are in only one plane.

Do you see why this property suggests that if the legs of a chair are not exactly the same length that you may be able to rest the chair on three legs, but not on four?



In the figure, there are three points, A, B, and C in a plane. The line through points A and B and the line through points B and C are drawn. The dotted lines are drawn so that they contain two points of the plane of A, B, and C. Each dotted line contains a point of the line through A and B, and a point of the line through B and C. The dotted lines are contained in the plane. The set of points represented by the dotted lines are contained in the plane. The plane which contains A, B, and C can now be described. It is the set of all points which are on lines containing two points of the figure consisting of the lines through A and B and through B and C.

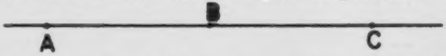
Exercises 1-1

1. How many different lines may contain:
   a. One certain point?
   b. A certain pair of points?
2. How many different planes may contain:
   a. One certain point?
   b. A certain pair of points?
   c. A certain set of three points?
3. How many lines can be drawn through four points, a pair of them at a time, if the points lie:
   a. In the same plane?
   b. Not in the same plane?
4. Explain the following: If two different lines intersect, one and only one plane contains both lines.

1-2. Names and Symbols

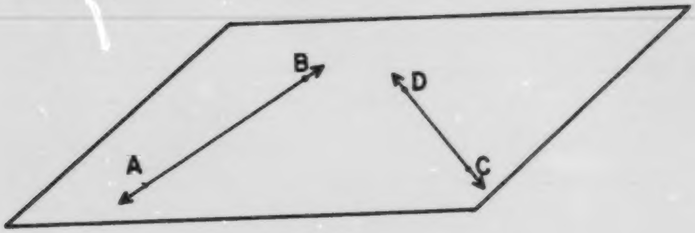
It is customary to assign a letter to a point and thereafter to say "point A" or "point B" according to the letter assigned.

A line may be represented in two ways, like this or simply like this . The drawing suggests all the points of the line, not just those that can be indicated on the page.

If you wish to call attention to several points on a line, you can do it in this way: 

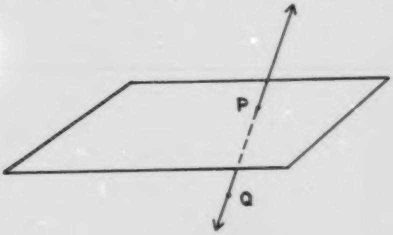
and the line may be called "line AB." A symbol for this same line is $\overleftrightarrow{AB}$. Other names for the above line are "line AC" or "line BC." The corresponding symbols would be $\overleftrightarrow{AC}$ or $\overleftrightarrow{BC}$.

Notice how frequently the word "represent" appears in these explanations. A point is merely "represented" by a dot because as long as the dot mark is visible, it has size. But a point, in geometry, has no size. Also lines drawn with chalk are rather wide, wavy, and generally irregular. Are actual geometric lines like this? Recall that "line" for us means "straight line." A drawing of a line by a very sharp pencil on very smooth paper is more like our idea of a line, but its imperfections will appear under a magnifying glass. Thus, by a dot you merely indicate the position of a point. A drawing of a line merely represents the line. The drawing is not the actual line. It is not wrong to draw a line free-hand (without a ruler or straight edge) but you should be reasonably careful in doing so. If you want to make a drawing of a plane on a piece of paper, you can use a diamond-shaped figure. Points of a plane are indicated in the same way as points of a line.



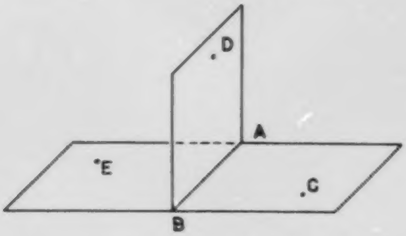
In the above figure, A, B, C, and D are considered to be points in the plane shown. A line is drawn through point A and point B. Another line is drawn through point C and point D. According to Property 2, $\overleftrightarrow{AB}$ and $\overleftrightarrow{CD}$ lie in the plane.

It is possible that a line might pierce or "puncture" a plane. A picture of this situation may appear thus:



The dotted portion of $\overleftrightarrow{PQ}$ would be hidden from view if the part of the plane represented were the upper surface of some object such as the card table.

Once again, you can see that the drawing only "represents" the situation.



How could you use Property 3 to identify the planes of the figure above? Property 3 states, "Any three points not all on the same line are in exactly one plane." Note that point A, point B, and point D are "not all on the same line." To show that apparently two names for the same plane may be given, it might be possible to write plane ABC = plane ABE if you are certain of what the equal sign means as used here.

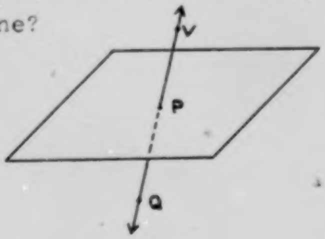
The notion of "set" will be helpful in explaining what is meant by "equal" when applied this way.

1. Plane ABC is a set of points.
2. Plane ABE is a set of points.
3. Points A, B, C, E, and others not indicated are in plane ABC and are also in plane ABE.

In fact, all elements of plane ABC, (a set of points) and elements of plane ABE (a set of points) seem to be contained in both sets (planes). You can say, "Two sets are equal if and only if they contain the same elements." According to this, plane ABC = plane ABE. In other words, you can say set M is equal to set N if M and N are two names for the same set.

Exercises 1-2

1. In the figure below, is point V a point of $\overleftrightarrow{PQ}$? Is point Q an element of the plane? Is V? How many points of $\overleftrightarrow{PQ}$ are elements of the plane?



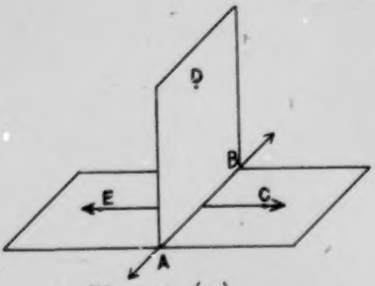


Figure (a)

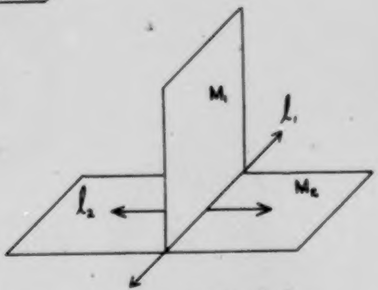


Figure (b)

2. Figure (b) is a copy of figure (a), except for labeling. Two different lines may be denoted as ℓ_1 and ℓ_2. The small numbers are called "subscripts." Plane ABD in figure (a) corresponds to M_1 in figure (b). $\overleftrightarrow{AB}$ in figure (a) corresponds to ℓ_1 in figure (b).

In the left-hand column are listed parts of figure (a). Match these with parts of figure (b) listed in the right-hand column:

Parts of Figure (a)

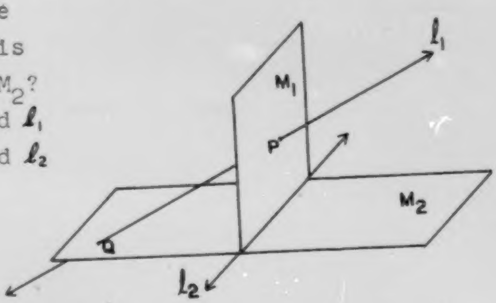
1. $\overleftrightarrow{EC}$
2. Plane ABC
3. Plane ABD
4. Plane EBA
5. $\overleftrightarrow{AB}$
6. The intersection of plane ABC and plane ABD

Parts of Figure (b)

a. ℓ_1
b. ℓ_2
c. M_1
d. M_2

Does the second column suggest an advantage of the subscript way of labeling?

3. In the figure at the right, (a) does ℓ_1 pierce M_1? (b) Also M_2? (c) Is ℓ_1 the only line through P and Q? (d) What is the intersection of M_1 and M_2? (e) Is ℓ_1 in M_2? (f) Would ℓ_1 meet ℓ_2? (g) Are ℓ_1 and ℓ_2 in the same plane?



1-3. Intersection of Sets

Now some useful and important ideas about sets shall be introduced.

Let set A = {1, 2, 3, 4, 5, 6, 7, 8, 9}
Let set B = {2, 4, 6, 8, 10, 12, 14, 16}

Let set C be the set of those elements which are in set A and are also in set B. You can write set C = {2, 4, 6, 8}. Set C is called the intersection of set A and set B.

Let set R be the set of pupils with red hair.
Let set S be the set of pupils who can swim.

It might happen that an element of set R (a pupil with red hair) might be an element of set S (a pupil who can swim). In fact, there may be no such common elements or there may be several.

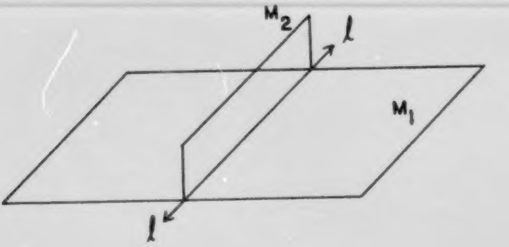
In any case, the set of red-headed swimmers is the intersection of set S and set R.

A set with no elements in it is called the "empty set." Thus, if there are no red-headed swimmers, then the intersection of set S and set R is the empty set.

The symbol $\cap$ is used to mean "intersection", that is, $E \cap F$ means "the intersection of set E and set F". Thus, referring to the sets mentioned previously, you can write:

$A \cap B$ is {2, 4, 6, 8}

$R \cap S$ is the set of red-headed swimmers



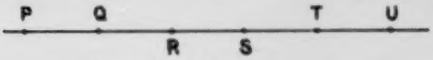
In the figure, line ℓ seems to be in M_1 and also in M_2. Every point in M_1 which is also in M_2 seems to be on the line ℓ. Thus the following statement seems to be true: $M_1 \cap M_2 = \ell$.

Some people talk about intersections in a slightly different way than it has been discussed here. When we say the intersection of two sets is empty, they say that the two sets do not intersect. When we say the intersection of two sets is not empty, they say that the two sets do intersect. The ideas are the same but the language is a bit different.

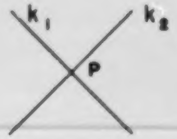
Exercises 1-4

1. Write the set whose members are:
    a. The whole numbers greater than 17 and less than 23
    b. The cities over 100,000 in population in your state
    c. The members of the class less than 4 years old
2. Write three elements of each of the following sets:
    a. The odd whole numbers
    b. The whole numbers divisible by 5

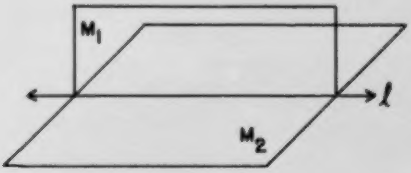
c. The set of points on the line below, some of which are labeled in the figure:



3. Give the elements of the intersections of the following pairs of sets:
   a. The whole numbers 2 through 12 and the whole numbers 9 through 20
   b. The set of points on line k_1 and the set of points on line k_2



   c. The set of points on plane M_1 and the set of points on plane M_2



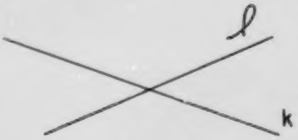
4. Let $S = \{4, 8, 10, 12, 15, 20, 23\}$
   $T = \{2, 7, 10, 13, 15, 21, 23\}$
   Find $S \cap T$.
5. Think of the top, bottom, and sides of a chalk box as sets of points.
   a. What is the intersection of two sides that meet?
   b. What is the intersection of the top and bottom?
6. Explain why "intersection" has the closure property and is both commutative and associative. In other words, if X, Y, and Z are sets, explain why:
   a. $X \cap Y$ is a set.
   b. $X \cap Y = Y \cap X$.
   c. $(X \cap Y) \cap Z = X \cap (Y \cap Z)$.

1-4. Intersections of Lines and Planes

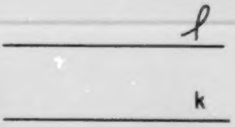
Two Lines

The possible intersections of two different lines may be described in three cases. Figures are drawn to represent the three cases.

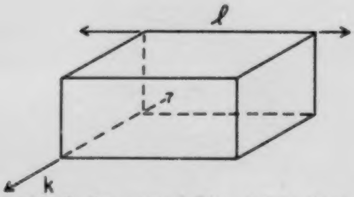
Case 1. ℓ and k intersect, or $\ell \cap k$ is not the empty set. ℓ and k cannot contain the same two points. Why?



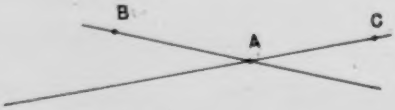
Case 2. ℓ and k do not intersect and are in the same plane. $\ell \cap k$ is the empty set, and ℓ and k are in the same plane. ℓ and k are said to be parallel.



Case 3. ℓ and k do not intersect and are not in the same plane. $\ell \cap k$ is the empty set. It is said that ℓ and k are skew lines.



In Problem 4 of section 1-1 you were asked to explain why two lines lie in the same plane if they intersect. In the figure above are shown two lines which intersect in point A. B is a point on one of the lines and C a point on the other. By Property 3, there is exactly one plane which contains A, B, and C.



By Property 2, $\overleftrightarrow{AB}$ is in this plane.

By Property 2, $\overleftrightarrow{AC}$ is in this plane.

There is exactly one plane which contains the two lines.

A Line and a Plane

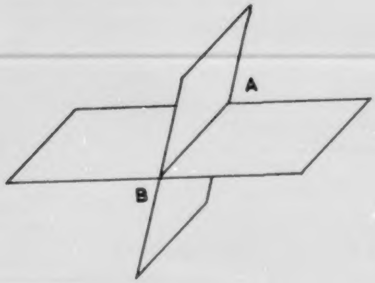
Again the possible intersections of a line and a plane may be described in 3 cases. Let M represent a plane and ℓ represent a line.

Case 1. M contains ℓ .
$M \cap \ell$ is the line ℓ .

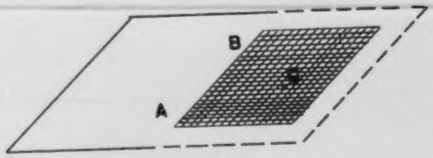
Case 2. M and ℓ intersect in exactly one point P.
$M \cap \ell$ is a point P.

Case 3. M and ℓ do not intersect.
$M \cap \ell$ is the empty set.
(Sometimes we say ℓ is parallel to M.)

Two Planes



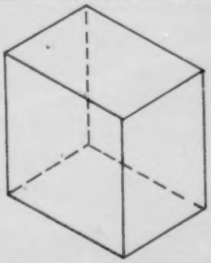
(a)



(b)

Let A and B be two points, each of which lies in two intersecting planes as in figure (a). From Property 2, the line $\overleftrightarrow{AB}$ must lie in each of the planes. Hence the intersection of the two planes contains a line. But if, as in figure (b), the intersection contains a point C not on the line $\overleftrightarrow{AB}$, then the two planes would be the same plane. By Property 3 there would be exactly one plane containing A, B, C. It can now be stated:

Property 4: If the intersection of two different planes is not empty, then the intersection is a line.



If the intersection of two planes is the empty set, then the planes are said to be parallel. Several examples of pairs of parallel planes are represented by certain walls of a room or a stack of shelves. Can you think of others?

In the discussion of the intersection of two different planes two cases have been considered. Let M and N denote the two planes.

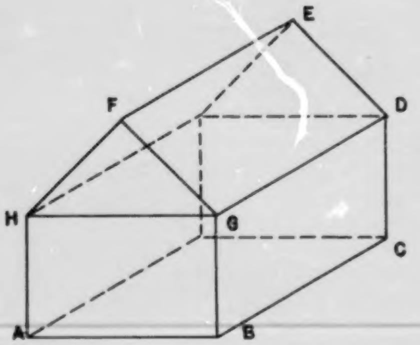
Case 1. $M \cap N$ is not empty. $M \cap N$ is a line.
Case 2. $M \cap N$ is empty. M and N are parallel.

Are there any other cases? Why?

Exercises 1-4

1. Describe two pairs of skew lines.
2. On your paper, label three points A, B, and C so that AB is not AC. Draw $\overleftrightarrow{AB}$ and $\overleftrightarrow{AC}$. What is $\overleftrightarrow{AB} \cap \overleftrightarrow{AC}$?
3. Carefully fold a piece of paper in half. Does the fold suggest a line? Stand the folded paper up on a table (or desk) so that the fold does not touch the table. Do the table top and the folded paper suggest three planes? Is any point in all three planes? What is the intersection of all three planes? Are any two of the planes parallel?
4. Stand the folded paper up on a table with one end of the fold touching the table. Are three planes suggested? Is any point in all three planes? What is the intersection of the three planes?
5. Hold the folded paper so that just the fold is on the table top. Are three planes suggested? Is any point in all three planes? What is the intersection of the three planes?
6. In each of the situations of Exercises 3, 4, and 5, consider only the line of the fold and the plane of the table top. What are the intersections of this line and this plane? You should have three answers, one for each of 3, 4, and 5.
7. Consider three different lines in a plane. How many points would there be with each point on at least two of the lines? Draw four figures showing how 0, 1, 2, or 3 might have been your answer. Why couldn't your answer have been 4 points.

8. Consider this sketch of a house.



Eight points have been labeled on the figure. Think of the lines and planes suggested by the figure. Name lines by a pair of points and planes by three points. Name:

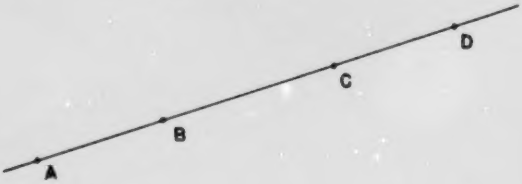
a. A pair of parallel planes.
b. A pair of planes whose intersection is a line.
c. Three planes that intersect in a point.
d. Three planes that intersect in a line.
e. A line and a plane whose intersection is empty.
f. A pair of parallel lines.
g. A pair of skew lines.
h. Three lines that intersect in a point.
i. Four planes that have exactly one point in common.

1-5. Segments and Union of Sets.

Consider three points A, B, and C not on the same line. Do you say that any one of them is between the other two? No, you usually do not. When it is said that a point P is between points A and B, there is a line containing A, B, and P.

Think of two different points A and B. The set of points consisting of A, B, and all points P between A and B is called the segment AB. A and B are called the endpoints. The segment with endpoints A and B is named by $\overline{AB}$. Another name for this segment is $\overline{BA}$.

Every segment has exactly two endpoints. As suggested above, each segment contains points other than its endpoints. Sometimes a segment is called a line segment.



In the figure above, three of the segments are named $\overline{AB}$, $\overline{CB}$, and $\overline{AC}$. What is the intersection of $\overline{AC}$ and $\overline{BD}$?

Not only may you talk about intersection of sets, but you may also find it convenient to talk about the union of sets. The word "union" suggests uniting or combining two sets into a new set. The union of two sets consists of those objects which belong to at least one of the two sets. For example, in the figure above, the union of $\overline{AB}$ and $\overline{BC}$ consists of all points of $\overline{AB}$, together with all points of $\overline{BC}$, that is, the segment $\overline{AC}$.

The symbol $\cup$ is used to mean "union". That is, $X \cup Y$ means "the union of set X and set Y". Suppose that set X is the set of numbers $\{1, 2, 3, 4\}$ and set Y is the set of numbers $\{2, 4, 6, 8, 10\}$. $X \cup Y$ is $\{1, 2, 3, 4, 6, 8, 10\}$. In the union of two sets, do not think of an element which occurs in both sets as appearing twice in the union.

Again, let us think of the set of all pupils who have red hair and the set of all pupils who can swim. You may think:

Let set R be the set of pupils with red hair.
Let set S be the set of pupils who can swim.

Then $R \cup S$ is the set of all pupils who either have red hair (whether or not they can swim) or who can swim (whether or not they have red hair).

Exercises 1-5

1. Draw a horizontal line. Label four points on it P, Q, R, and S in that order from left to right. Name two segments:
    a. Whose intersection is a segment.
    b. Whose intersection is a point.
    c. Whose intersection is empty.

d. Whose union is not a segment.

2. Draw a line. Label three points of the line A, B, and C with B between A and C.
   - a. What is $\overline{AB} \cap \overline{BC}$?
   - b. What is $\overline{AC} \cap \overline{BC}$?
   - c. What is $\overline{AB} \cup \overline{BC}$?
   - d. What is $\overline{AB} \cup \overline{AC}$?
3. Draw a segment. Label its endpoints X and Y. Is there a pair of points of $\overleftrightarrow{XY}$ with Y between them?

4. Draw two segments $\overline{AB}$ and $\overline{CD}$ for which $\overline{AB} \cap \overline{CD}$ is empty but $\overleftrightarrow{AB} \cap \overleftrightarrow{CD}$ is one point.
5. Draw two segments $\overline{PQ}$ and $\overline{RS}$ for which $\overline{PQ} \cap \overline{RS}$ is empty, but $\overleftrightarrow{PQ}$ is $\overleftrightarrow{RS}$.
6. Let A and B be two points. Is it true that there is exactly one segment containing A and B? Draw a figure explaining this problem and your answer.
7. Draw a vertical line ℓ. Label A and B two points to the right of ℓ. Label C a point to the left of ℓ. Is $\overline{AB} \cap \ell$ empty? Is $\overline{AC} \cap \ell$ empty?
8. Explain why "union" has the closure property and is both commutative and associative. In other words, if X, Y, and Z are sets, explain why:
   - a. $X \cup Y$ is a set
   - b. $X \cup Y = Y \cup X$
   - c. $(X \cup Y) \cup Z = X \cup (Y \cup Z)$
9. Show that for every set X you will have:

$$X \cup X = X$$

1-6. Separations

In this section a very important idea will be considered--the idea of separation. You shall see this idea applied in three different cases.

Let A and B be any two points of space not in a plane M. If the intersection of $\overline{AB}$ and M is empty, then A and B are on the same side of M. The set of all points on the same side of M as A, we call a half-space. If the intersection of $\overline{AB}$ and M is not empty, then A and B are on opposite sides of M. The set of all points on the opposite side of M from A is another half-space. Thus

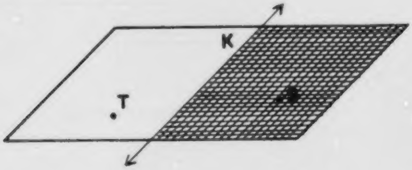
Any plane M *separates space into two half-spaces*.

The plane M is called the boundary of each of the half-spaces.

Now consider only a plane M, and a line ℓ in that plane. Let A and B be two points not on the line. Then either $\overline{AB} \cap \ell$ is the empty set, in which case A and B are on the same side of ; or $\overline{AB} \cap \ell$ is not the empty set, in which case A and B are on opposite sides of ℓ. Similarly then:

Any line ℓ *of a plane* M *separates the plane into two half-planes*.

In the figure below the S-side of line k is shaded and the T-side of k is not shaded.



Now consider a line ℓ. How would you define a half-line? Can you say anything about segments in this definition as you did in defining half-planes and half-spaces? What would the boundary be? Is the boundary a set of points?

Our third case should now be clear. Can you state it?

It is important to note that these three cases are almost alike. They deal with the same idea in different situations. Thus:

Case 1. Any plane separates space into two *half-spaces*.

Case 2. Any line of a plane separates the plane into two *half-planes*.

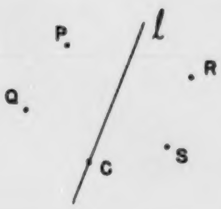
Case 3. Any point of a line separates the line into two *half-lines*.

There is one other definition that is useful. A *ray* is a half-line together with its endpoint. A ray has one endpoint. A ray without its endpoint is a *half-line*. A ray is usually drawn like this, ⟶. If A is the endpoint of a ray and B is another point of the ray, the ray is denoted by $\overrightarrow{AB}$. Note that $\overrightarrow{BA}$ is not $\overrightarrow{AB}$. The term ray is used in the same sense in which it is used in "ray of light".

In everyday language, you sometimes do not distinguish between lines, rays, and segments. In geometry you should distinguish between them. A "line of sight" really refers to a ray. You do not describe somebody as in your line of sight if he is behind you. The right field foul line in baseball really refers to a segment and a ray. The segment extends from home plate through first base to the ball park fence. It stops at the fence. The ray starts on the ground and goes up the fence. What happens to a home run ball after it leaves the park makes no difference to the play in a major league game.

Exercises 1-6

1. In the figure at the right, is the R-side of ℓ the same as the S-side of ℓ ? Is it the same as the Q-side? Are the intersections of ℓ and $\overline{PQ}$, ℓ and $\overline{RS}$ empty? Are the intersections of ℓ and $\overline{QS}$, ℓ and $\overline{PR}$ empty? Considering the sides of ℓ , are the previous two answers what you would expect?



2. Draw a line containing points A and B. What is $\overrightarrow{AB} \cap \overrightarrow{BA}$? What is the set of points not in $\overrightarrow{AB}$?
3. Draw a horizontal line. Label four points of it A, B, C and D in that order from left to right.
   Name two rays (using these points for their description):
   a. Whose union is the line.
   b. Whose union is not the line, but contains A, B, C, D.
   c. Whose union does not contain A.
   d. Whose intersection is a point.
   e. Whose intersection is empty.
4. Does a segment separate a plane? Does a line separate space?
5. The idea of a plane separating space is related to the idea of the surface of a box separating the inside from the outside. If P is a point on the inside and Q a point on the outside of a box, does $\overline{PQ}$ intersect the surface?
6. Explain how the union of two half-planes might be a plane.

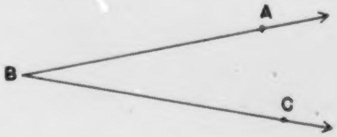
7. If A and B are points on the same side of the plane M (in space), must $\overrightarrow{AB} \cap M$ be empty? Can $\overrightarrow{AB} \cap M$ be empty?

1-7. Angles and Triangles

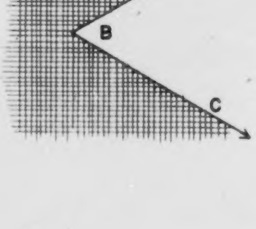
Angles

Some of the most important ideas of geometry deal with angles and triangles. An angle is a set of points consisting of two rays with an endpoint in common and not both on the same straight line. Let us say this in another way. Let BA and BC be two rays such that A, B, and C are not all on the same line. Then the set of points consisting of all the points of BA together with all the points of BC is called the angle ABC. An angle is the union of two rays. The point B is called the vertex of the angle. The rays BA and BC are called the rays (or sometimes the sides) of the angle. An angle has exactly one vertex and exactly two rays.

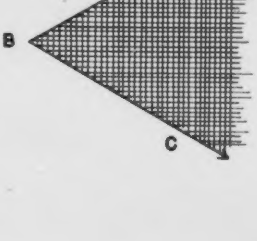
An angle is drawn in the figure below. You will recall from section 1-2 that what is really meant is "a representation of an angle is drawn". Three points of the angle are labeled so that the angle is read "angle ABC" and may be written as, "$\angle$ABC." The letter of the vertex is always listed in the middle. Therefore, $\angle$ABC is $\angle$CBA. Note that in labeling this angle the order of A and C does not matter, but B must be in the middle. Is $\angle$ABC the same as $\angle$BAC (not drawn)?



From the figure it looks as if the angle ABC separates the plane containing it. It is true that the angle does separate the plane. The two pieces into which the angle separates the plane look somewhat different. They look like:



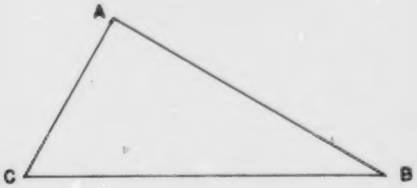
and



The piece on the right is called the *interior* of the angle and the one on the left the *exterior*. The interior of the $\angle$ABC can be defined as the intersection of the A-side of the line $\overleftrightarrow{BC}$ and the C-side of the line $\overleftrightarrow{AB}$. It is the intersection of two half planes and does *not* include the angle. The exterior is the set of all points of the plane not on the angle or in the interior.

Triangles

Let A, B, and C be three points not all on the same straight line. The *triangle* ABC, written as $\triangle$ ABC, is the union of $\overline{AB}$, $\overline{AC}$, and $\overline{BC}$. You will recall that the union of two sets consists of all of the elements of the one set together with all the elements of the other. The $\triangle$ ABC can be defined in another way. The triangle ABC is the set of points consisting of A, B, and C, and all points of $\overleftrightarrow{AB}$ between A and B, all points of $\overleftrightarrow{AC}$ between A and C, and all points of $\overleftrightarrow{BC}$ between B and C. The points A, B, and C are the vertices of $\triangle$ ABC. The word "vertices" is used when referring to more than one vertex. Triangle ABC is represented in the figure.



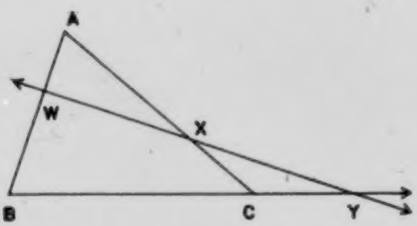
Angles of a Triangle

You can speak of $\overline{AB}$, $\overline{AC}$, and $\overline{BC}$ as the *sides* of the triangle. You can speak of $\angle$ABC, $\angle$ACB, and $\angle$BAC as the *angles* of the triangle. These are the angles determined by the triangle. The sides of the triangle are contained in the triangle but the angles of a triangle are not.

Note that a triangle is a set of points in exactly one plane. Every point of the triangle ABC is in the plane ABC. $\triangle$ ABC separates the plane in which it lies. The $\triangle$ ABC has an interior and an exterior. The interior is the intersection of the interiors of the three angles of the triangle. The exterior is the set of all points of plane ABC not on $\triangle$ ABC or in the interior of $\triangle$ ABC.

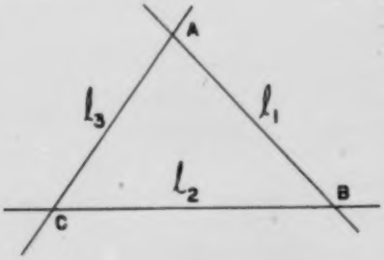
Exercises 1-7

1. Label three points A, B, and C not all on the same line. Draw $\overleftrightarrow{AB}$, $\overleftrightarrow{AC}$, and $\overleftrightarrow{BC}$. (a) Shade the C-side of $\overleftrightarrow{AB}$. Shade the A-side of $\overleftrightarrow{BC}$. What set is now doubly shaded? (b) Shade the B-side of $\overleftrightarrow{AC}$. What set is now triply shaded?
2. Draw a triangle ABC.
    - a. In the triangle, what is $\overline{AB} \cap \overline{AC}$?
    - b. Does the triangle contain any rays or half-lines?
    - c. In the drawing extend $\overline{AB}$ in both directions to obtain $\overleftrightarrow{AB}$. What is $\overline{AB} \cap \overleftrightarrow{AB}$?
    - d. What is $\overleftrightarrow{AB} \cap \triangle ABC$?
3. Refer to the figure on the right.



    - a. What is $\overline{YW} \cap \triangle ABC$?
    - b. Name the four triangles in the figure.
    - c. Which of the labeled points, if any, are in the interior of any of the triangles?
    - d. Which of the labeled points, if any, are in the exterior of any of the triangles?
    - e. Name a point on the same side of $\overleftrightarrow{WY}$ as C and one on the opposite side.
4. Draw a figure like that of Exercise 4.
    - a. Label a point P not in the interior of any of the triangles.
    - b. Label a point Q inside two of the triangles.
    - c. If possible, label a point R in the interior of $\triangle$ ABC but not in the interior of any other of the triangles.
5. If possible, make sketches in which the intersection of a line and a triangle is:
    - a. The empty set
    - b. A set of two elements
    - c. A set of one element
    - d. A set of exactly three elements
6. If possible, make sketches in which the intersection of two triangles is:
    - a. The empty set
    - b. Exactly two points
    - c. Exactly four points
    - d. Exactly five points

7. In the figure, what are the following:
   a. $\angle ABC \cap \overleftrightarrow{AC}$
   b. $\triangle ABC \cap \overleftrightarrow{AB}$
   c. $\ell_1 \cap \angle ACB$
   d. $\overline{AB} \cap \ell_2$
   e. $\angle BCA \cap \angle ACB$
   f. $\overline{BC} \cap \angle ABC$
   g. $\overrightarrow{BC} \cap \angle ACB$
   h. $\angle ABC \cap \triangle ABC$



8. In a plane if two triangles have a side of each in common, must their interiors intersect? If three triangles have a side of each in common, must some two of their interiors intersect?

9. Draw $\angle ABC$. Label points X and Y in the interior and P and Q in the exterior.
   a. Must every point of $\overline{XY}$ be in the interior?
   b. Is every point of $\overline{PQ}$ in the exterior?
   c. Can you find points R and S in the exterior so that $\overline{RS} \cap \angle ABC$ is not empty?
   d. Can $\overline{XP} \cap \angle ABC$ be empty?

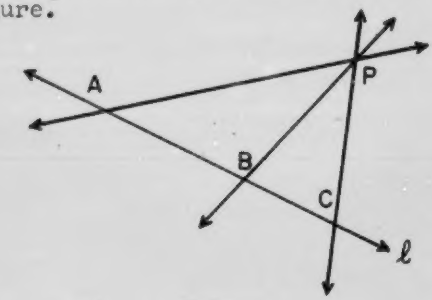
1-8. One-to-One Correspondences

The idea of "one-to-one correspondence" was used in talking about counting numbers. This idea is also useful in geometry. By "one-to-one correspondence" is meant the matching of each member of a certain set with a corresponding member of another set. Before this idea is used in geometry, let us review our previous experience.

Consider the set of counting numbers less than eleven. Let us form two sets from these numbers. Set A, containing the odd numbers: {1, 3, 5, 7, 9} and set B, containing the even numbers: {2, 4, 6, 8, 10}. There is a one-to-one correspondence between set A and set B because every odd number can be matched with an even number.

Exercises 1-8

1. Is there a one-to-one correspondence between the states of the United States and the U. S. cities of over 1,000,000 in population? Why? Does the fact that Nevada contains no city of over a million population show that no such correspondence exists?
2. Show that there is a one-to-one correspondence between the set of even whole numbers and the set of odd whole numbers.
3. If set R is in one-to-one correspondence with set S and set S with set T, is set R in one-to-one correspondence with Set T? Why?
4. Establish a one-to-one correspondence between the set of even whole numbers and the set of whole numbers.
5. Follow the given directions in making a drawing somewhat like the given figure.



Questions are scattered throughout the directions. Answer the questions as you go along.

(a) Draw a line and label it ℓ .

(b) Choose a point not on line ℓ and label it P.

(c) Mark some point A on line ℓ .

(d) Draw line $\overleftrightarrow{PA}$.

  (1) Is $\overrightarrow{PA} \cap \ell$ equal to the empty set?

  (2) Does the intersection set of PA and ℓ have only one element? Why?

(e) Choose two other points, B and C on ℓ . Draw $\overleftrightarrow{PB}$ and $\overleftrightarrow{PC}$.

  (1) Through each additional point marked on ℓ can you draw a line that also goes through point P?

  (2) Let all lines which intersect ℓ and pass through P be the elements of a set called K. How many elements of K have been drawn up to now?

(3) Does each indicated element of K contain a point on ℓ ?

(4) Can each indicated element of K be matched with an indicated point on ℓ ?

(5) Do you think that, if more elements of K were drawn and more points on ℓ were marked, each element of K could be matched with a corresponding element of ℓ ?

6. Describe a one-to-one correspondence between the points A, B, and C which determine a triangle and the sides of the triangle. Can you do this in more than one way?

7. Draw a triangle with vertices A, B, and C. Label a point P in the interior of $\triangle$ ABC. Let H be the set of all rays having P as an endpoint. The elements of H are in the plane of $\triangle$ ABC. Draw several rays of H. Can you observe a one-to-one correspondence between H and $\triangle$ ABC? For very point of $\triangle$ ABC is there exactly one ray of H containing it? For every ray of H is there exactly one point of $\triangle$ ABC on such ray?

8. Draw an angle XYZ with the vertex at Y. Draw the segment $\overline{XZ}$. Think of K as a set of rays in plane XYZ with common endpoint at Y. K is the set of all such rays which do not contain points in the exterior of $\angle$XYZ. $\overrightarrow{YX}$ and $\overrightarrow{YZ}$ are two of the many elements of K. Draw another element of K. Does it intersect $\overline{XZ}$? For each element of K will there be one such matching point of $\overline{XZ}$? Label D a point of $\overrightarrow{YX}$ and E a point of $\overrightarrow{YZ}$. Draw $\overline{DE}$. Is there a similar one-to-one correspondence between the set of points of $\overline{XZ}$ and the set of points of $\overline{DE}$?

1-9. Simple Closed Curves

A curve is a set of points which can be represented by a pencil drawing made without lifting the pencil off the paper; it being understood that in the process part of the drawing may be retraced.

Segments and triangles are examples of curves you have already studied. Curves may or may not contain portions that are straight. In everyday language the term "curve" is used in this same sense. When a baseball pitcher throws a curve, the ball seems to go straight for a while and then "breaks" or "curves".

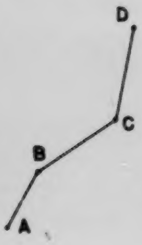


Fig. a

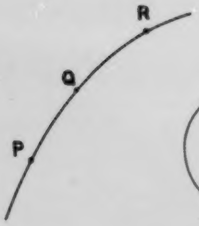


Fig. b

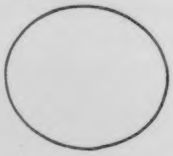


Fig. c

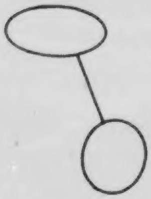


Fig. d

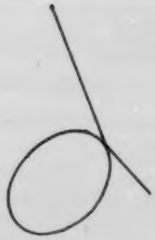


Fig. e

One important type of curve is called a broken-line curve. It is the union of several line segments; that is, it consists of all the points on several line segments. Fig. a represents a broken-line curve. A, B, C, and D are indicated as points on the curve. It is also said that the curve contains or passes through these points. Figures b to i also represent curves. In Fig. b, points P, Q, and R are indicated on the curve. Of course, the curve can be thought of as containing many points other than P, Q, and R.

A curve is said to be a simple closed curve if it can be represented by a figure drawn in the following manner. The drawing starts and stops at the same point. Otherwise, no point is touched twice and the drawing instrument is not removed from the paper in the process. Figures c, g, h, and i represent simple closed curves. The other figures of this section do not. Figure j represents two simple closed curves. The boundary of a state like Iowa or Utah on an ordinary map represents a simple closed curve.

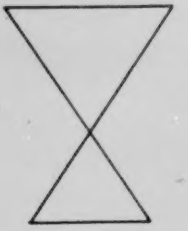


Fig. f

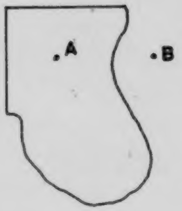


Fig. g

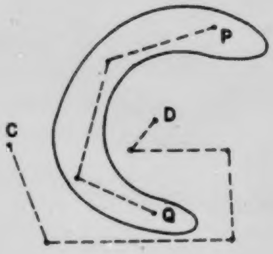


Fig. h

The examples that have been mentioned, including that of a triangle, suggest a very important property of simple closed curves. Each simple closed curve has an interior and an exterior in the plane. Furthermore, any curve at all containing a point in the interior and a point in the exterior must intersect the simple closed curve. As an example, consider any curve containing A and B of Fig. g and lying in the plane. Also any two points in the interior (or any two points in the exterior) may be joined by a broken-line curve which does not intersect the simple closed curve. Fig. h indicates this. A simple closed curve is the boundary of its interior and also of its exterior.

The interior of a simple closed curve is called a region. There are other types of sets in the plane which are also regions. In Fig. j, the portion of the plane between the two simple closed curves is called a region. Usually a region (as a set of points) does not include its boundary.

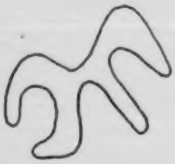


Fig. i

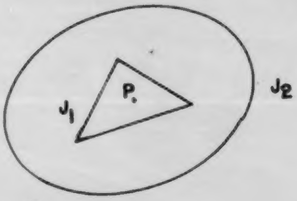
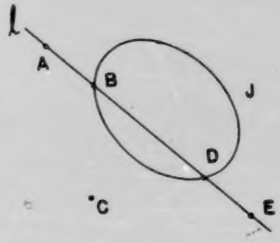


Fig. j

Consider figure j. The simple closed curve (represented by) J_1 is in the interior of the simple closed curve J_2. You may obtain a natural one-to-one correspondence between J_1 and J_2 as follows. Consider a point such as P in the interior of J_1. Consider the set of rays with endpoint at P. Each such ray intersects each of J_1 and J_2 in a single point. Furthermore, each point of J_1 and each point of J_2 is on one such ray. A point of J_1 corresponds to a point of J_2 if the two points are on the same ray from P. Note that this procedure using rays would not determine a one-to-one correspondence if one of the simple closed curves were like that in Fig. i.

Exercises 1-9

1. Draw a figure representing two simple closed curves whose intersection is exactly two points. How many simple closed curves are represented in your figure?
2. In Fig. j, describe the region between the curves in terms of intersection, interior and exterior.
3. Draw two triangles whose intersection is a side of each. Is the union of the other sides of both triangles a simple closed curve? How many simple closed curves are represented in your figure?
4. In a map of the United States, does the union of the boundaries of Colorado and Arizona represent a simple closed curve?
5. The line ℓ and the simple closed curve J are as shown in the figure.
   a. What is $J \cap \ell$?
   b. Draw a figure and shade the intersection of the interior of J and the C-side of ℓ .
   c. Describe in terms of rays the set of points on ℓ not in the interior of J.



6. Draw two simple closed curves, one in the interior of the other such that, for no point P do the rays from P establish a one-to-one correspondence between the two curves. Consider Fig. i.
7. Draw two simple closed curves whose interiors intersect in three different regions.
8. Explain why the intersection of a simple closed curve and a line cannot contain exactly three points if the curve crosses the line when it intersects it.

Chapter 2

MEASUREMENT

2-1. Counting and Measuring

Such common questions as "How many people went to the ball game?" or "How much meat shall I buy?" or How fast can a jet travel?" have answers which are alike in one respect: they all involve numbers. Some of these answers are found by counting, while others are found by measuring.

The question, "How many?" indicates that you are thinking of a set of objects and wish to have some measure of how many there are in the set. Such a set is called a discrete set. Questions as, "How much?" "How long?" "How fast?" etc., are used to describe something thought of as all in one piece, without any breaks. Such a set is called a continuous set. Sets of people, houses, or animals are discrete sets; a rope, a road, or a flagpole are all thought of as being continuous since they are like line segments; you can count a number of line segments but you cannot count the number of all points on a line segment. A blackboard and a pasture may be thought of as sets of points enclosed by simple closed curves, and as being continuous. Such sets of points are not counted, they are "measured".

Properties of Continuous Quantities

The sizes of line segments and of simple closed curves can be compared without actually measuring them. To compare segments, $\overline{AB}$ and $\overline{CD}$ lay the edge of a piece of paper along $\overline{CD}$ and mark points C and D. Place the edge of the paper along $\overline{AB}$ with point C on point A. If D is between A and B, $\overline{AB}$ is longer than $\overline{CD}$. If D falls on B, the segments are the same length; these segments are said to be congruent. If B is between C and D, $\overline{CD}$ is longer than $\overline{AB}$.

The union of the set of all points which lie on a simple closed curve or are contained in the interior of the simple closed curve will be called a closed region. To compare the sizes of two closed regions, one may be cut and placed over the other.

Certain properties of geometric continuous quantities are assumed:

Motion Property. A geometric figure may be moved without changing its size or shape.

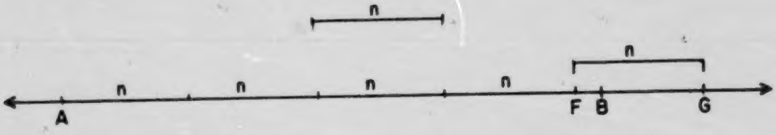
Comparison Property. Two continuous geometric figures or sets, of the same kind may be compared to determine whether they have the same size, or which one is the larger.

Subdivision Property. A geometric continuous figure or set may be subdivided.

Subdivision and Measurement.

The subdivision property of a continuous quantity is the basis for the process called measuring. If a segment $\overline{AB}$ is subdivided by a point C so that $\overline{AC} = \overline{CB}$, then $\overline{AC}$ is one half of $\overline{AB}$. If two points D and E are placed on $\overline{AB}$ so as to subdivide $\overline{AB}$ into three equal parts, how does $\overline{AB}$ compare with $\overline{AD}$? with $\overline{AE}$?

$\overline{AB}$ can be subdivided so as to compare the length of one line segment with the length of another line segment. Suppose a segment is chosen of any length less than the length of $\overline{AB}$; call the length of the segment "n".



Beginning at point A in the figure, above a segment of length n is marked off 4 times so that $\overline{AF}$ is of length 4n. The symbol "4n" means "four times as long as the segment of length "n". It is said that the length of $\overline{AB}$ is approximately equal to 4n. A symbol for the words "approximately equal to" is a wavy equals sign like this, "$\approx$". You can state that the length of $\overline{AB} \approx 4n$. Notice how these symbols are used:

4 is the measure,
n is the unit of measurement,
4n is the length.

Subdividing Units of Measurement

You have seen that the sizes of continuous quantities can be found by measuring, and that for this a unit of measurement must be used. The unit of measurement must have two charistics:

1. It must be of the same nature as the thing to be measured--a line segment to measure a line segment, a closed region to measure a closed region, and so on.
2. It must be possible to move the unit, or to copy it accurately, so that it can be used to subdivide the thing that is measured.

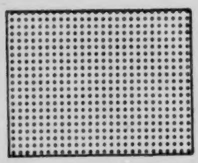
In measuring a rectangular region it is assumed that the result does not depend on how the rectangular units are placed on the large figure. The assumption that the area obtained does not depend on the way the units are placed on a figure can be tested. Measure the same rectangular closed region with the same unit, but place the units on the region in different ways. If you actually try this "experiment" you should get the same results, within the limits of experimental error. The assumption that a continuous quantity can be measured by covering it with units in any way found convenient is very important. If, in this experiment, different results were obtained, then a single rectangle would seem to have two different sizes. To agree with our assumption that continuous quantities of the same kind can be compared, the fact that the whole rectangle must have greater area than any part of it, is accepted.

To measure this rectangular closed region a smaller rectangular closed region may be used as a unit.

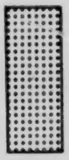
Exercises 2-1

1. Draw a triangle, a circle, a square and a rectangle. If each of these is used to measure the area of a sheet of notepaper
   (a) Would any of the figures be hard to use as a unit of measurement? If so, what is the reason?
   (b) Would the measure be the same number for any two of the figures?
2. Use the closed region B as a unit to compare the areas of the two closed regions.

A

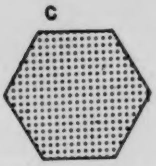


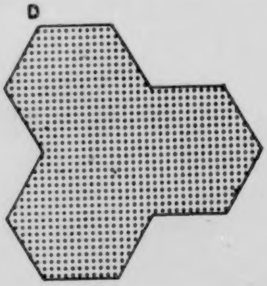
B



The size of closed region A is approximately ______ times the size of B.

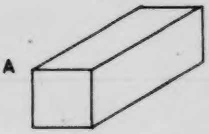
3. (a) Compare the length of curve D with the length of curve C.

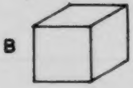




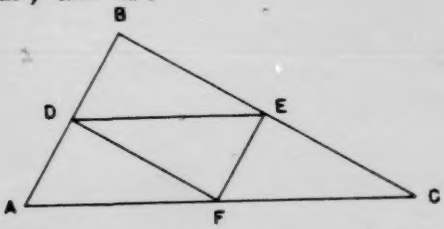
(b) Compare the size of the closed region D with the size of the closed region C.

4. Subdivide solid A into parts the size of box B. The solid box A is about ______ times as large as the solid B.





5. Draw a triangle ABC. On each side of the triangle place a point which subdivides the side into two congruent segments. Name these points D, E, and F, as shown in the sketch. Draw segments $\overline{DE}$, $\overline{EF}$, and $\overline{DF}$.



(a) Use a strip of paper to determine which segments are congruent. (You should find three sets, with three segments in each set.)

(b) There are all simple closed curves in this figure. Some are triangles and some are four-sided figures. Name all the simple closed curves.

(c) Copy triangle EFC by tracing it, and compare the area of the closed region EFC with the areas of the other triangular closed regions.

(d) What sets of closed regions are the same size? (One set should have four members and two sets should each have 3 members.)

6. Could the space inside a box be measured by finding how many marbles of a certain size it will hold? How would the measurement be stated? What different shaped object would be better to measure this space?

2-2. Standard Units of Length

Two people using different units to measure the same thing may well have difficulty comparing results; at least some agreement is advisable. When a unit is agreed upon and used by a large number of people, it is a standard unit.

"Linear" means "having the nature of a line". All measurement of line segments is called linear measurement.

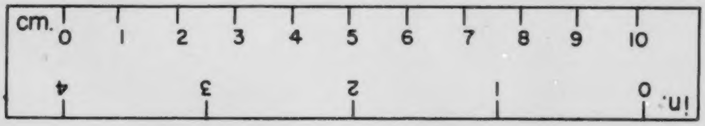
In history disagreements about linear units became so common that a group of French scientists called a conference of representatives from many countries to meet with them and establish an international set of measures. This group developed the metric system which discarded the old units and based all of their units on the distance from the North Pole to the equator. The meter is the basic unit of length in the metric system. (The meter was planned to be one ten-millionth of the distance from the North Pole to the Equator, but recently an international congress of scientists defined the meter in relation to light waves). The metric system is used by all scientists in the world and is in common use in all countries except those in which English is the main language spoken.

The National Bureau of Standards in Washington has an accurate copy of the meter, a bar made of platinum and iridium, since this metal changes very little in length under different atmospheric conditions. Congress has passed a law that tells what part of this bar shall be the official yard in the United States. This bar is *the standard unit of length* with which all standard units of length in this country are compared. If a manufacturer wants to make measuring instruments, his products must match the official standard.

The Ruler and the Number Line

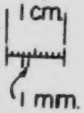
Measuring segments may be considered the same as comparing the segment that is to be measured with a number line. Actually, an ordinary ruler represents a part of a number line on which the unit is an inch; this part of the number line starts at zero and goes to 12.

In the first part of this section it was stated that the *meter* is the basic unit of length in the *Metric System*. However, it is frequently convenient to use another unit to measure lengths which are smaller than one meter. This unit is the centimeter which is $\frac{1}{100}$ of a meter. Many commercially made rulers have one edge labeled in inches and the other edge labeled in centimeters. The following picture will help you visualize the size of one centimeter and its approximate relationship to the inch.



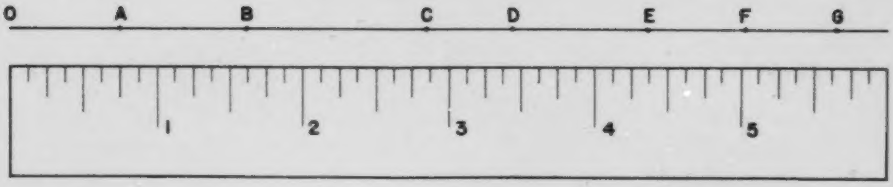
$$1 \text{ centimeter} = \frac{1}{100} \text{ of a meter}$$
$$1 \text{ meter} = 100 \text{ cm.}$$

Under many laboratory conditions it is frequently necessary to use a unit of measure smaller than the centimeter. For these purposes scientists use a unit which is $\frac{1}{10}$ of a centimeter, called a millimeter. Notice that $\frac{1}{10} \times \frac{1}{100}$ of a meter is $\frac{1}{1000}$ of a meter. The prefix "milli" means one-thousandth. The following picture will help fix the size of a 1 millimeter unit in our minds.



$$1 \text{ millimeter} = \frac{1}{10} \text{ of a centimeter} = \frac{1}{1000} \text{ of a meter.}$$
$$1 \text{ meter} = 1000 \text{ mm.}$$

Exercises 2-2

1. 

What point on the ruler is below each of the points, A through G, on the line segment?

2. How long is:

(a) $\overline{AB}$? (b) $\overline{AC}$? (c) $\overline{CD}$? (d) $\overline{DF}$? (e) $\overline{FG}$?

3. (a) Draw a segment 5 inches long and divide it into sections each $\frac{5}{8}$ inch long.

(b) Divide 5 by $\frac{5}{8}$.

(c) Is there any relation between parts (a) and (b) of this problem?

4. Convert the following measurements to the unit indicated; write the steps in detail and state answers in decimal form.

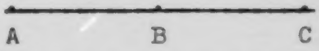
(a) 17 cm. to mm. (b) 3.4 m. to cm.

(c) 48 mm. to cm. (d) 357 mm. to m.

(e) 93 cm. to m. (f) 9.1 m. to mm.

Meaning of Linear Measurement

A line segment may be considered as a set of points. When a line segment is measured in inches, 1 inch units are "placed side by side" on the line segment so that two consecutive units have exactly one point in common. The segment $\overline{AC}$ is the union of the sets determined by

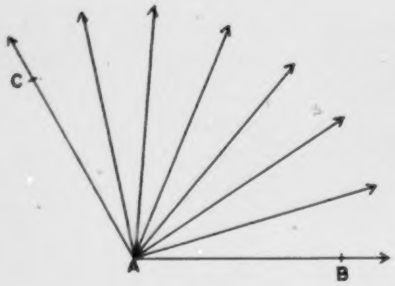


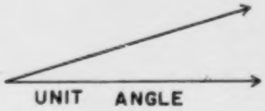
$\overline{AB}$ and $\overline{BC}$. If a line segment is exactly 6 inches long this merely states that it is the union of six unit segments for each 1 inch long. This use of "union" is a conceptual basis for linear measurement; in actual practice the notion that approximation is involved is paramount. A similar use of union of sets will appear later in connection with area and volume.

2-3. Measurement of Angles

An angle determines three sets of points in the plane, the set of points in the interior of the angle, the set of points in the exterior of the angle, and the set of points on the angle itself. A point P is in the interior of angle BAC if it is on the same side of line $\overleftrightarrow{AB}$ as point C, and on the same side of line $\overleftrightarrow{AC}$ as point B. Any point in the plane which is not a point on the angle and not a point in the interior is a point in the exterior of the angle.

As has been stated previously, to measure anything you must use a unit of the same nature as the thing measured. To measure an angle, an angle is chosen as the unit of measure. In the following sketch, the interior of angle BAC is subdivided so that angles are formed which are exactly like the unit angle shown. The measure of $\angle$BAC, for this unit angle, is 7; its measurement is 7 unit angles.





Standard Unit for Angles

Just as there are standard units for measuring a line segment (inch, foot, yard, millimeter, centimeter, meter) so are there standard units for measuring an angle. Start with a line and a point on it. Through that point draw 179 rays so that they, together with the two rays on the given line, determine 180 congruent angles. These angles, together with their interiors, make a half-plane together with the line which determines it. The rays are numbered in order from 0 to 180, forming a scale. To each ray corresponds a number and conversely. In the sketch below, only the ray corresponding to 0 and every tenth ray thereafter is drawn. One of these 180 congruent angles is selected as the standard unit; its measurement is called a degree and its measure in degrees, is 1.

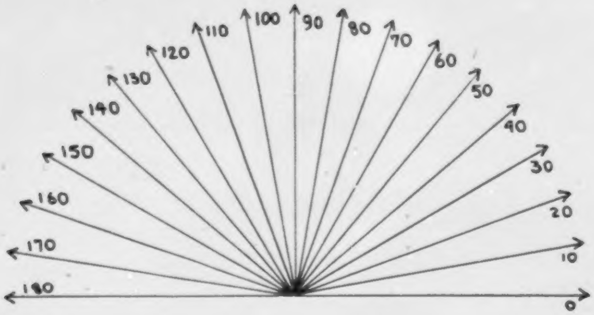


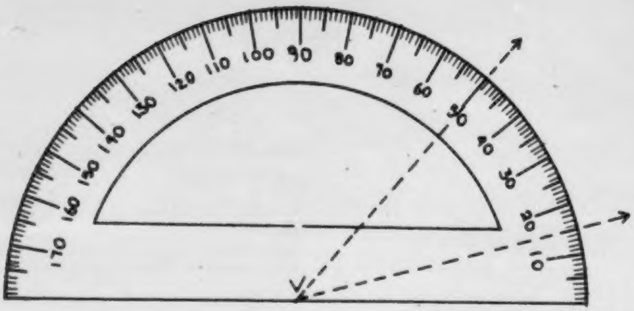
Figure 2-3

A scale like this can be used to measure an angle. Place the angle on the scale with one side of the angle on the ray that corresponds to zero and the vertex of the angle at the intersection of the rays. Then the number which corresponds to that ray along which the other side of the angle falls is the measure of the angle, in degrees. The size or measurement of the angle is that number of degrees.

The symbol for "degree" is "$^{\circ}$" so that thirty-five degrees may be written "35°".

The Protractor. Since placing an angle on a scale is inconvenient, an instrument called a protractor is usually used. Then the scale can be placed on the angle, rather than the angle on the scale.

Consider this drawing of a protractor and think of the rays from point V. In the drawing, two of the rays are shown in dotted lines, while on the actual instrument a portion of each ray is shown on the curved part.



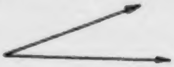
To measure an angle with the protractor, place the protractor on the angle so that point V is on the vertex of the angle and the ray which corresponds to zero on the protractor lies on one side of the angle. Then the number which corresponds to the protractor ray which is on the other side of the angle is the measure, in degrees, of the angle.

An actual protractor has two scales, one of which you read from right to left and the other from left to right.

Sets of Angles. Angles may be separated into sets according to their measurement or measures.

An angle of 90 degrees is called a right angle.

An angle whose measurement is less than 90 degrees is called an acute angle.



An angle whose measure is more than 90 and less than 180 is called an obtuse angle.

Perpendicular Lines

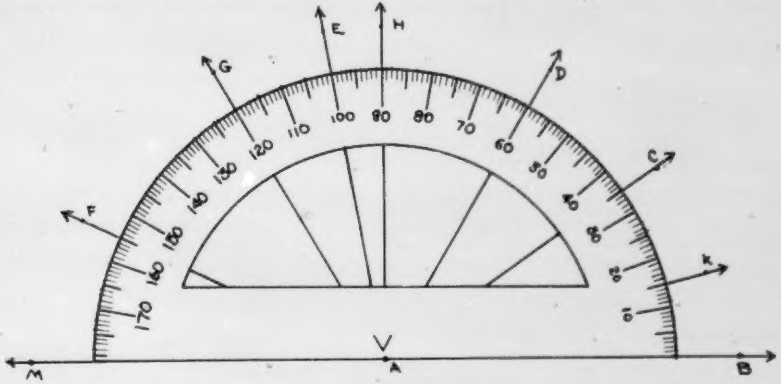
When two lines intersect, they are perpendicular if one of the angles determined by the lines is a right angle. Line segments are perpendicular if the lines on them are perpendicular.

In similar fashion a line segment may be perpendicular to a ray or to a line. The symbol for "perpendicular" is "$\perp$".

Exercises 2-3

1. In the next drawing a protractor is shown placed on a figure with several rays drawn from point A. Find the measure, in degrees, of each of the angles.

(a) ∠BAK (b) ∠BAC (c) ∠BAD (d) ∠BAH (e) ∠BAE
(f) ∠MAF (g) ∠GAM (h) ∠MAC (i) ∠DAE (j) ∠CAG
(k) ∠KAF (l) ∠HAF



2. Describe how to use a protractor to construct an angle whose
   (a) measurement is 20°.
   (b) measure, in degrees, is 61.

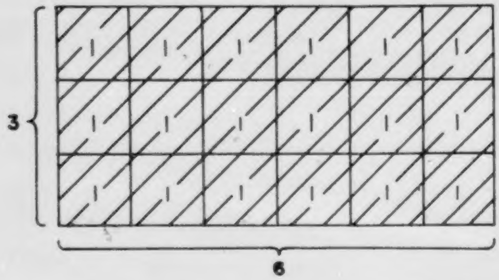
2-4. Area

The most familiar of the simple closed curves, the rectangle, is a four-sided figure (in a plane) which has a right angle at each of its four corners. Under this definition is a square a rectangle?

If the length and width of a rectangle are denoted by ℓ linear units and w linear units, respectively, then the perimeter of a rectangle = $(2\ell + 2w)$ linear units. Note particularly that ℓ and w are measures so that the measure of the width in the linear unit being used = w and the width of the rectangle is equal to w linear units.

Opposite sides have equal lengths and intersecting sides are perpendicular.

If a rectangle is 6 units long and 3 units wide, then the rectangular closed region can just be covered by unit square areas as shown. Is the large rectangle



the union of the eighteen small rectangles? With respect to the "union" concept note the similarity between the measurement of this rectangular closed region and the measurement of a line segment.

The measure of the area is, then, by definition, the number of unit square closed regions. There is an easier way to get the number than by counting them.

Using ℓ and w , as defined above,

area of a rectangle = $\ell \cdot w$ square units.

Exercises 2-4

1. A farmer has had a rectangular garden for a number of years. He knows that the length of wire fence around it is 500 feet and has found by experience that he uses a 100-pound bag of fertilizer on it each year. One spring he decides to enlarge his garden so it will be twice as long and twice as wide. Since the old fence is worn out anyway he throws it away. He then goes to the hardware store and orders 1000 feet of fencing and 2 hundred-pound bags of fertilizer. Is this order reasonable? Why or why not?
2. (a) A rectangle is 3 units long and 2 units wide. If another rectangle is twice as long but has the same width, how do the areas of the two rectangles compare? Draw a figure illustrating the answer. Do the same if the new rectangle has the same length as the original one but twice the width.
   (b) Does the reasoning used in part (a) depend on the particular measures, 3 and 2? If not, write a statement

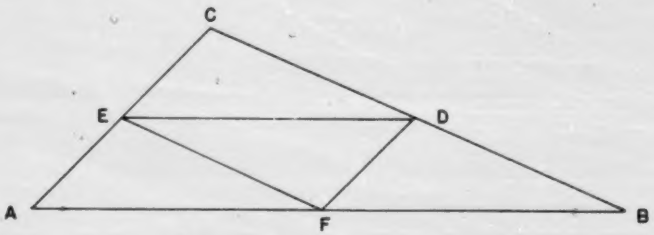
telling the effect on the area of doubling the length of any rectangle. On doubling the width.

3. (a) If a rectangle has length and width of 3 units and 2 units, what is the effect on the area of doubling both length and width? Draw a figure to illustrate your conclusion. Use mathematical notation to describe this situation for a general rectangle.

   (b) In the rectangles of part (a) compare the two perimeters. Write a statement telling the effect on the perimeter of a rectangle of doubling both length and width. Is this true for any rectangle?

4. A farmer found that it took 240 feet of fence to go around his rectangular farmyard. He noticed that one of the sides is 40 feet long. How long is each of the other sides. Let x stand for the number of feet in the width and write a number sentence describing this problem.

5. In the figure below, D, E, F are the midpoints of the sides. Let a stand for the number of units in the length of segment $\overline{AF}$.



Label with an a all the segments which have the same length as $\overline{AF}$. Similarly, let b be the number of units in the length of $\overline{AE}$ and c the number of units in the length of $\overline{BD}$. Write number sentences for the number of units in the perimeters of ΔCED and ΔABC.

In the previous sets of exercises English units were used in finding perimeters and areas of closed regions. Now consider the use of some of the metric units. The metric units for linear measurements were the meter, the centimeter and the millimeter. The corresponding metric units for area measurement are square closed regions having edges which measure 1 meter, 1 centimeter, and 1 millimeter respectively.

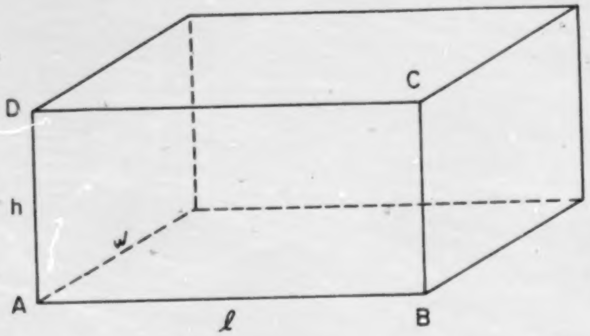
6. How many square millimeters are there in a square centimeter?
7. How many square centimeters are there in a square meter?
8. How many square millimeters are there in a square meter?
9. A rug is 2 meters by 3 meters. Find its perimeter and area.
10. The floor of a room is in the shape of a rectangle. The length and width are measured as 4 meters and 3 meters. There is a closet 1 meter long and 1 meter wide built in one corner. Find the floor area of the room (outside the closet).

2-5. Rectangular Prism

A figure shaped like that shown in the next diagram will be referred to as a rectangular prism; most rooms have this shape. The flat sides of the prism, each lying in a plane, are called its faces, e.g., ABCD. Note that each of the six faces is a closed rectangular region. For each face of the prism there is just one other face that does not meet it; these are called opposite faces. Opposite faces actually lie in parallel planes and are congruent. The set of all points lying on the six faces constitute the prism. The prism has an interior; if you walk in a room which has the shape of a rectangular prism you are walking in the interior of the prism.

Since two planes which intersect must intersect in a line, two faces which are not opposite must intersect in points that lie on a line, actually the points of a line segment. These segments are called the edges of the prism. How many edges are there on a rectangular prism? Some of these edges have the same length. How many different lengths could there be among the edges? There are on the prism certain points where three faces intersect, or what amounts to the same thing, where three edges intersect. These points are called vertices (singular: vertex) of the prism. How many vertices are there?

As you noticed, parallel edges have the same length, hence, there can be at most three different lengths. In the figure below, the number of units in the lengths of three edges have been marked as ℓ , w, and h; these letters were chosen since the lengths of



the edges in the three possible directions are often called the *length*, *width*, and *height* of the prism. Since all the faces are rectangular closed regions, their areas can be found readily; the sum of the areas of all the faces is called the *surface area* of the rectangular prism.

Volume

The term *rectangular solid* will refer to the set of points which is the union of a rectangular prism *and* its interior. To measure this rectangular solid, you find its *volume*. This volume will be referred to as the volume of the prism; the last phrase is a figure of speech since actually a prism does not have volume, though it does have area. When you measure a volume, you pick out some convenient unit of volume and see how many such units are necessary to make up the solid.

The term "volume of the interior of a rectangular prism" can be used in the same manner as the term "area of the interior of a rectangle" was used. The volume of the interior of a rectangular prism is equal to the volume of the corresponding rectangular solid.

What must you know in order to find the volume of a rectangular prism? This procedure of finding the volume of a prism from the area of the base and height without needing to know the exact shape of the base will be used again when other prisms and cylinders are considered.

If all the edges of a rectangular prism are known, the area of the base can be found readily, then the volume. If a rectangular prism a units by b units by c units, has as base the face whose area is $a \times b$ square units, its volume would be $c \times (a \times b)$ cubic units. Now stand the prism on end so that the b by c face is the base. Using previously stated ideas the volume of the solid should be written as $a \times (b \times c)$ cubic units. Resting the prism on its third face, the volume is $b \times (a \times c)$ cubic units. Since it is the same solid in different positions we have

$$c \times (a \times b) = a \times (b \times c) = b \times (a \times c).$$

Exercises 2-5

1. A room is 15 feet long, 12 feet wide, and 9 feet high; there are five windows in the walls, each 3 feet wide and 6 feet high.
   (a) How many asphalt tiles, each 12 inches by 12 inches, will be necessary to tile the floor?
   (b) How many tiles will be necessary if each is 6 inches by 6 inches?
   (c) How much wall surface is there, not counting the windows? In finding your answer does it matter where the windows are placed?
   (d) How many quarts of paint are necessary to paint these walls if a pint of paint will cover 66 square feet?
2. A trunk is 3 feet long, 18 inches wide, and 2 feet high. The edges are all reinforced with strips of brass. How much brass stripping is necessary? Express the answer in inches, in feet, and in yards.
3. A cube is a rectangular prism for which all the edges are congruent so all the faces are square closed regions. How many square inches of wood are needed to make a covered cubical box with edges of 18 inches each? How many square feet?
4. Let ℓ, w, h stand for the numbers of units in the length, width, and height of a rectangular prism.
   (a) Write a number sentence telling how to find the number, S, of square units in the surface area.

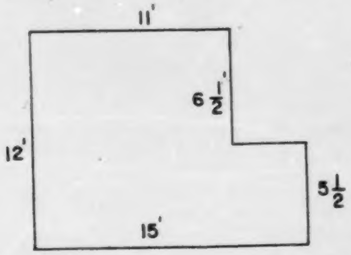
(b) Write a number sentence telling how to get the total number, E, of units of length in all the edges.

5. Write a statement telling how to find the number of cubic units in a rectangular prism if the number of square units in the base and the number of units in the height are known.
6. A rectangular box is h units high and has a base whose area is B square units. Write a number sentence showing how to get the number, V, of cubic units of volume in its interior if the numbers h and B are known. Notice that this is just a mathematical sentence for the statement you made in Problem 5.
7. If ℓ, w, h, stand for the number of units in the length, width, and height of a rectangular prism, write a number sentence telling how to find the number of cubic units, V, of volume in the interior of the solid.
8. How many cubic inches are there in a cubic foot? How many cubic feet are in a cubic yard? Show how to obtain these conclusions.
9. (a) In a 3-inch cube (a cube each edge of which is 3 inches), what is the volume of the cube?
   (b) Is it larger or smaller than 3 cubic inches? Be very careful not to confuse the volume of a 3-inch cube with a measured volume of 3 cubic inches.
   (c) If ℓ is the number of inches in the edge of a cube, is it always true that the volume of an ℓ inch cube is greater than ℓ cubic inches?
10. If a rectangular prism is 2 inches long and 1 inch wide, what is its height if the rectangular solid is to have a volume of 1 cubic inch?
11. Tell what will be the effect on the volume of a rectangular prism if all its measurements are tripled? What will be the effect if two are doubled and one tripled?
12. An iron rod is to be made with a square cross section 1 inch on a side. If a cubic foot of iron were molded into this shape, how long would the rod be? Express the answer in inches, in feet, and in yards.

13. If a building were a one-mile cube, that is a cube one mile on an edge would there be space for the whole population of the United States (about 176,000,000)? For the population of China (about 600,000,000)? Could the population of both countries together be accomodated? (Allow 10 cubic yards per person.)

14. The length, width, and height of a rectangular prism are measured as $10\frac{1}{2}$ inches, $5\frac{0}{2}$ inches, and $3\frac{1}{2}$ inches. Find the volume of its interior from these measurements.

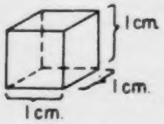
15. The floor plan of a room is as shown:



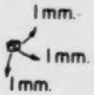
How many square feet of wall-to-wall carpeting would be necessary for the floor? What is the volume of the interior of the room if it is 9 feet high?

16. A pantry, the floor of which is 4 ft. by 5 ft., is 9 ft. high. It contains a deep freeze which is 2 ft. by 3 ft. by 7 ft. How many cubic feet of space are left in the room? Express the answer also in cubic yards.

In the preceding exercises volumes were calculated in terms of cubic yards, cubic feet and cubic inches. Some metric units for measuring volumes are the cubic meter, cubic centimeter and cubic millimeter. A cubic centimeter and a cubic millimeter are pictured below.



1 cubic centimeter

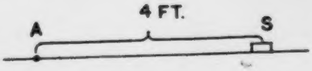


1 cubic millimeter

17. How many cubic millimeters in 1 cubic centimeter?
18. How many cubic centimeters in 1 cubic meter?
19. How many cubic millimeters in 1 cubic meter?
20. Draw a cube 3 cm. on each edge. Draw also a rectangular prism whose volume is 3 cubic centimeters. Which has the greater volume?
21. The length, width, and height of a rectangular prism are 2 meters, 3 meters, and 1 meter. Find its total surface area and its volume.
22. Suppose that in a room 5 meters long, 4 meters wide, and 3 meters high, a closet is built in one corner. This closet runs to the ceiling and has a base 1 meter on a side. Find the volume of the room without the closet.

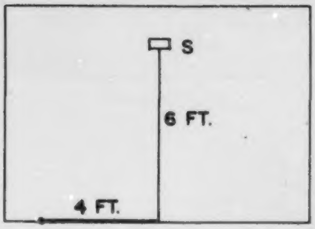
Dimension

Consider two flies side by side at a point A by the baseboard of a room. One of them is trying to direct the other to crawl to a lump of sugar which is also by the baseboard. What direction does he need to give?



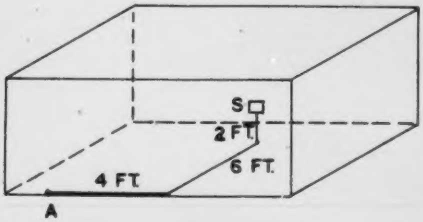
Presumably all he needs to say is "Follow the baseboard this way for four feet--you can't miss it!" The complete description of where the sugar is located by the baseboard can be given by one number and one direction. For this reason the edge of the room is called one-dimensional. Of course, the section of baseboard followed may be a single segment, or may turn one or more corners, so any segment or simple closed curve is one-dimensional.

If the lump of sugar S is somewhere out in the middle of the floor, this presents more of a problem to the directing fly. His friend cannot get there at all by following the baseboard. How can he give directions? One of the simpler ways would be as shown below.



"Follow the baseboard this way for four feet. Then turn to the left so you are headed perpendicular to the baseboard and crawl for six feet." In this case when the lump of sugar was interior to the rectangle it was found convenient to use two numbers and two directions parallel to an edge of the room to describe its location. For this reason the set interior to a rectangle (or any simple closed curve) is called two-dimensional.

If the lump of sugar is not on the floor at all but is somewhere else in the room, for example, suspended from the ceiling by a string, the problem of direction is harder still.



The directions then might go like this, "Crawl this way along the baseboard for four feet, then turn to the left and proceed along the floor perpendicular to the baseboard for six feet. You will then be directly under the sugar. To get to the sugar fly directly up for two feet". This time three numbers and three directions parallel to an edge of the room have been used to describe how to get to the point S, so the interior of the room (that is, the interior of a rectangular solid) is called three-dimensional.

On the basis of the above discussion what dimension would you want to give a point?

Roughly, the dimension of the set in which the fly moves shows how much freedom of motion he has. If he must stay in the one-dimensional set consisting of the floor's edge, he can only move one way or the other along this edge. If he may go anywhere in the two-dimensional set inside the rectangular edge he can crawl all over the floor. If he is merely confined to the three-dimensional set interior to the room, he can fly all over the room.

TABLES FOR REFERENCE

English Units	Metric Units
Measurements of Length	
12 in. = 1 ft.	10 millimeter (mm.) = 1 centimeter (cm.)
3 ft. = 1 yd.	100 cm. = 1 meter (m.)
16½ ft. = 1 rd.	1,000 m. = 1 kilometer (km.)
320 rd. = 1 mi.	
5280 ft. = 1 mi.	
Measurements of Surface	
144 sq. in. = 1 sq. ft.	100 sq. mm. = 1 sq. cm.
9 sq. ft. = 1 sq. yd.	10,000 sq. cm. = 1 sq. m.
160 sq. rd. = 1 acre	1,000,000 sq. m. = 1 sq. km.
43,560 sq. ft. = 1 acre	
640 acres = 1 sq. mi.	
Measurements of Volume	
1728 cu. in. = 1 cu. ft.	1,000 cu. mm. = 1 cc.
27 cu. ft. = 1 cu. yd.	1,000,000 cc. = 1 cu. m.

Metric and English Equivalents

1 in. = 2.54 cm.	1 cm. ≈ 0. 4 in.
1 yd. ≈ 0.9 m.	1 m. ≈ 1.1 yd.
1 mi. ≈ 1.6 km.	1 km. ≈ 0.62 mi.
1 lb. ≈ 0.45 kg.	1 m. ≈ 39.37 in.
1 qt. ≈ 0. 95 ℓ.	1 kg. ≈ 2.2 lb.
	1 ℓ. ≈ 1.05 qt.

Chapter 3

PARALLELOGRAMS AND TRIANGLES

3-1. Two Lines on a Plane

Figure 3-1a shows that the intersection of lines ℓ_1 and ℓ_2 is point A. Rays on ℓ_2 are $\overrightarrow{AB}$ and $\overrightarrow{AC}$. (Recall that $\overrightarrow{AB}$ means the ray with A as endpoint and containing point B). Rays on ℓ_1 are $\overrightarrow{AE}$ and $\overrightarrow{AD}$.

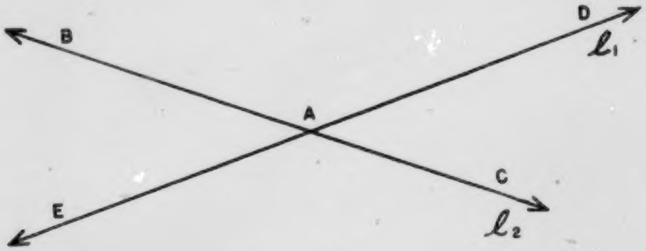


Figure 3-1a

Observe that there are four angles formed by the rays in Figure 3-1a: angle BAE, angle BAD, angle CAD, and angle CAE. These angles shall be spoken of as angles formed by ℓ_1 and ℓ_2. We shall speak of two intersecting lines as determining four angles.

Angles CAD and DAB have a common ray, $\overrightarrow{AD}$, and a common vertex, point A. Any two angles which have a common ray, a common vertex, and whose interiors have no point in common are called *adjacent angles*. Thus, angles CAD and DAB are adjacent angles.

Angles BAD and EAC are not adjacent angles, but they are both formed by rays on the two lines ℓ_1 and ℓ_2. When two lines intersect, a pair of non-adjacent angles formed by these lines are called *vertical angles*. Note that here "vertical" is not associated with "horizontal." Thus, angles BAE and CAD are a pair of vertical angles.

Two intersecting lines separate a plane into four regions. Each of these four regions is the interior of an angle. The interior of each angle is the intersection of two half-planes. The intersection of the shaded half-plane determined by ℓ_1 and the shaded half-plane determined by ℓ_2 is shown in Figure 3-1b. The intersection of these two half-planes is the interior of angle BAE. The interior of angle CAD is the intersection of the two remaining unshaded half-planes. These two angles, angle BAE and angle CAD, are vertical angles.

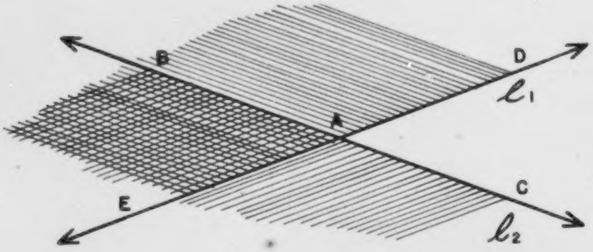


Figure 3-1b

If the sum of the measures in degrees of two angles is 180, the angles are called supplementary angles. In Figure 3-1b, the shaded angles BAE and BAD are supplementary. These supplementary angles are also adjacent angles. However, supplementary angles need not be adjacent angles.

To indicate the number of units in an angle, the symbol "m" followed by the name of the angle enclosed in parentheses will be used. For example, $m(\angle ABC)$ means the number of units in angle ABC. Any angle can be used as a unit of measure, but in this chapter the degree will be used as the standard unit. Thus when you write $m(\angle ABC = 40$, angle ABC is a 40 degree (40°) angle. Note that since $m(\angle ABC)$ is a number, $m(\angle ABC) = 40$, not "$m(\angle ABC) = 40^\circ$."

Exercises 3-1

Use Figure 3-1c in answering 1 through 3.

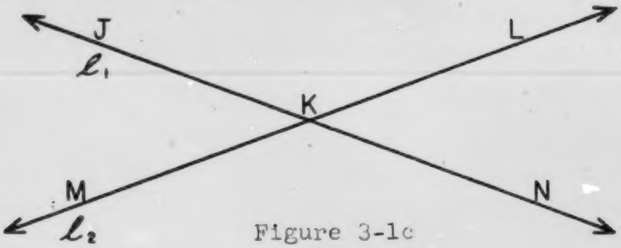
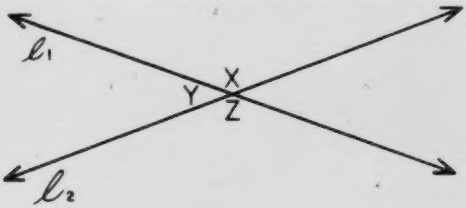


Figure 3-1c

1. (a) Name the angles adjacent to $\angle JKM$.
   (b) Name the angles adjacent to $\angle JKL$.
2. (a) Name the angle which with $\angle JKM$ completes a pair of vertical angles.
   (b) When two lines intersect in a point. how many pairs of vertical angles are formed?
3. (a) Use a protractor to find the measures of the vertical angles, $\angle JKM$ and $\angle LKN$.
   (b) Draw sets of two intersecting lines as in Figure 3-1c. Vary the size of the angles between the lines. With a protractor find the measures of each pair of vertical angles.
   (c) What appears to be true concerning the measures of a pair of vertical angles?

Property 1: When two lines intersect, the two angles in each pair of vertical angles which are formed have equal measure, or are congruent.

4. The following figure is similar to Figure 3-1c. Let x, y, and z represent the angles LKJ, JKM, and NKM respectively. The angles are indicated in this way in the figure.



Copy and complete the following statements:

(a) $m(\angle x) + m(\angle y) =$ __?__.

(b) If $m(\angle y)$ is known, how can you find $m(\angle x)$? How can you find $m(\angle z)$?

(c) Write your answer for part (b) in the form of a number sentence as is done in parts (a) and (b) by copying and completing the following:

$m(\angle x) =$ __?__ - __?__

$m(\angle z) =$ __?__ - __?__

(d) Write a number sentence to show the relation between $m(\angle x)$ and $m(\angle z)$.

3-2. Parallel Lines and Corresponding Angles

A line which intersects two or more lines in distinct points is called a transversal of those lines.

In Figure 3-2a, t is a transversal. From the vertex of angle f there is a ray which extends upward on line t. This ray contains a ray of angle b. Also, the interiors of angle b and angle f are on the same side of the transversal t. Angles placed in this way are called corresponding angles.

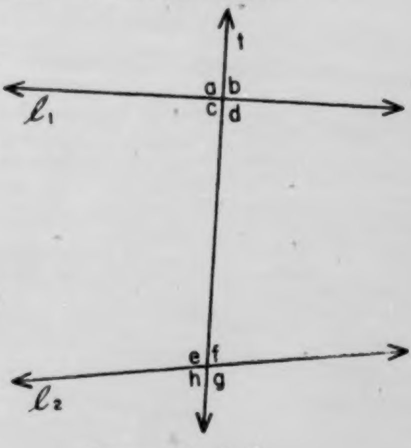


Figure 3-2a

If two lines r_1 and r_2, are cut by the transversal t, and lines r_1 and r_2 do not intersect, in the language of sets,

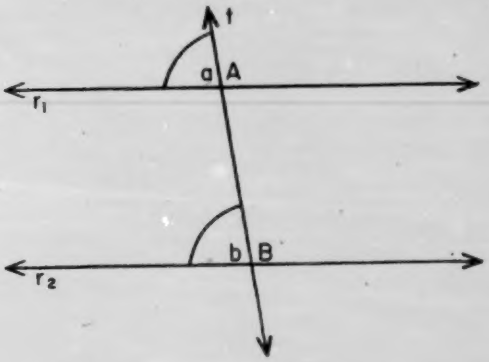


Figure 3-2b

$r_1 \cap t$ is not the empty set,
$r_2 \cap t$ is not the empty set,
$r_1 \cap r_2$ is the empty set.

That is; r_1 and r_2 are parallel. Neither r_1 nor r_2 is parallel to t.

Exercises 3-2

1. Use Figure 3-2a to answer Problem 1.
    (a) Name another pair of corresponding angles on the same side of the transversal as angles b and f.
    (b) How many pairs of corresponding angles are in Figure 3-2a?
    (c) If the measures of ∠ a and ∠ e are 90, what can you say about the measures of all the angles in Figure 3-2a?
    (d) If the measures of ∠ a and ∠ e are 90, are angles h and b supplementary angles? Explain.
2. In the following table, using fig. 2b, predict whether r_1 and r_2 intersect and, if so, where. Make a drawing to check your prediction

Measure of ∠ a in degrees	Measure of ∠ b in degrees	Intersection of r_1 and r_2
50	80	
50	50	
50	40	

3. Use Figure 3-2c. Copy the table and complete it.

Measure of x in degrees	Measure of p in degrees	ℓ_1 and ℓ_2		
		Are Parallel	Intersect above t	Intersect below t
(a) 120	140			
(b) 120	120			
(c) 120	90			
(d) 90	120			
(e) 90	90			
(f) 90	70			

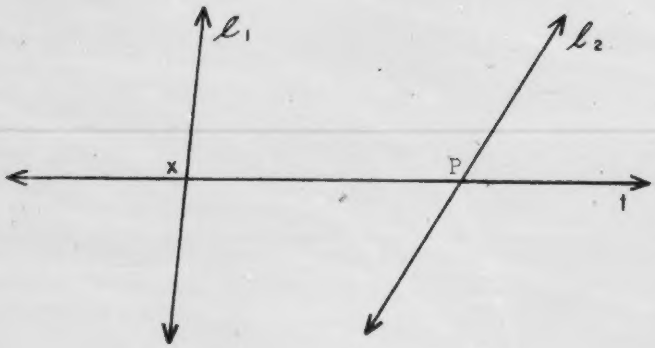


Figure 3-2c

As a result of these problems we are ready to state two properties.

Property 2: When, in the same plane, a transversal intersects two lines and a pair of corresponding angles have different measures, then the lines intersect.

Property 2a: When, in the same plane, a transversal intersects two lines and a pair of corresponding angles are congruent, then the lines are parallel.

3-3. Converses

Certain statements are written in the form, "If . . ., then . . .," for example,

(a) "If two angles are vertical angles, then the angles have the same measure."

Suppose a new statement is made by interchanging the "if" part and the "then" part. Thus,

(b) "If two angles have the same measure, then the angles are vertical angles." A statement obtained by such an interchange is called a converse statement. In the example above, (b) is called the converse of (a). Since such an interchange in (b) brings us back to (a), you can also call (a) the converse of (b).

If you interchange the parts of a true statement, will the new statement always be true?

1. "If Mary and Sue are sisters, then Mary and Sue are girls."

Converse of 1: "If Mary and Sue are girls, then Mary and Sue are sisters." Is the original statement true? Is the converse also true?

Now consider the next statement:

2. "If Lief is the son of Eric, then Eric is the father of Lief."

Converse of 2: "If Eric is the father of Lief, then Lief is the son of Eric." Is the original statement true? Is the converse true?

These two illustrations show that, if a statement is true, a converse obtained by interchanging the "if" part and the "then" part, may be true or may be false.

3. Is statement (a) above, dealing with vertical angles, true? Is the converse statement, (b), true? One cannot accept a converse statement of a true statement as always being true.

Exercises 3-3

1. Make a drawing for which a converse of statement (a) in Section 3-3 is not true. Must any two angles which have the same measure always be vertical angles?
2. For each of the following statements write "true" if the statement is always true; "false" if the statement is sometimes false.
   (a) If Blackie is a dog, then Blackie is a cocker spaniel.
   (b) If Robert is the tallest boy in his school, Robert is the tallest boy in his class.
   (c) If an animal is a horse, the animal has four legs.
3. Write a converse for each statement in Problem 2 and tell whether it is true or false.
4. Read the following statements. Write "true" if the statement is always true; "false" if the statement is sometimes false.
   (a) If a figure is a circle, then the figure is a simple closed curve.
   (b) If a figure is a simple closed curve composed of three line segments, then the figure is a triangle.

(c) If two angles are congruent, they are right angles.

(d) If two lines are parallel, then the lines have no point in common.

(e) If two angles are supplementary, they are adjacent.

(f) If two adjacent angles are both right angles, they are supplementary.

5. Write a converse statement for the following property.

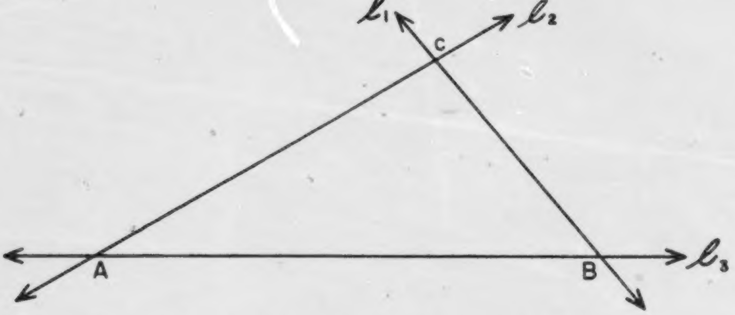
Property 2a: If, in the same plane, a transversal intersects two lines, and a pair of corresponding angles are congruent, then the lines are parallel.

Is the converse true or false?

6. Write a converse statement for Property 2: If, in the same plane, a transversal intersects two lines, and a pair of corresponding angles have different measures, then the lines intersect. Does the converse statement seem to be true or false?

7. Can a converse for a false statement be true? If so, can you give an example?

3-4. Triangles



In the drawing above, ℓ_1 is a transversal for ℓ_2 and ℓ_3; ℓ_2 is a transversal for ℓ_1 and ℓ_3; ℓ_3 is a transversal for ℓ_1 and ℓ_2. A, B, and C are three points and $\overline{AB}$, $\overline{BC}$, and $\overline{CA}$ are segments joining them in pairs. The union of three points not on the same line and the segments joining them in pairs is called a triangle. Our sketch, according to this definition, contains triangle ABC. The points A, B, and C are called the vertices of the triangle and the segments $\overline{AB}$, $\overline{BC}$, and $\overline{CA}$ are called the sides of the triangle.

Triangles that have at least two congruent sides are called *isosceles* triangles. If three sides are congruent the triangle is called *equilateral*. If no two sides are congruent the triangle is called *scalene*.

The union of $\overline{AB}$, $\overline{BC}$, and $\overline{AC}$ is called "triangle ABC." Notice that points C and B are the only points shared by angle A and side $\overline{BC}$. Angle A and side $\overline{BC}$ are said to be opposite each other because their intersection consists only of the endpoints of $\overline{BC}$. Is $\angle B$ opposite $\overline{AC}$? Why?

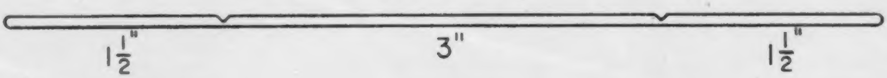
Property 3: In an isosceles triangle, the angles opposite congruent sides are congruent angles.

You can convince yourself that Property 3 is true by drawing isosceles and equilateral triangles, and by measuring the angles opposite the congruent sides.

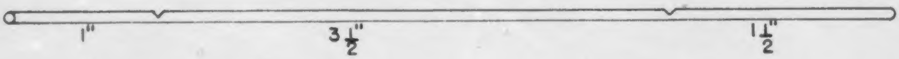
Similarly the converse of Property 3 can be checked by drawing triangles with a pair of angles, or three angles, congruent and measuring the sides opposite these angles.

Exercises 3-4

1. If a triangle is isosceles, is it also equilateral? Explain your answer.
2. If a triangle is equilateral, is it also isosceles? Explain your answer.
3. (a) Could a triangle be represented by folding the soda straw shown in this figure? Explain your answer.



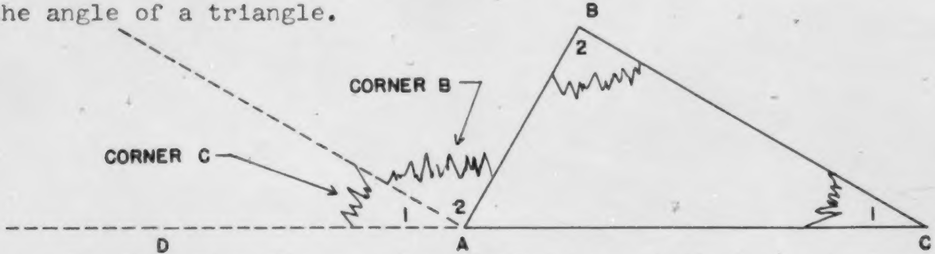
(b) Could a triangle be represented by folding the soda straw shown here? Explain your answer.



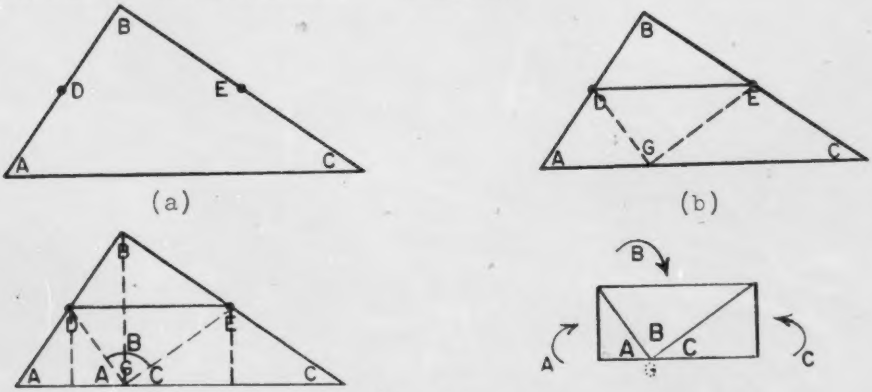
(c) State a property about the lengths of the sides of a triangle as suggested by your observations in parts (a) and (b).

3-5. Angles of a Triangle

The figure shows a property about the sum of the measures of the angle of a triangle.



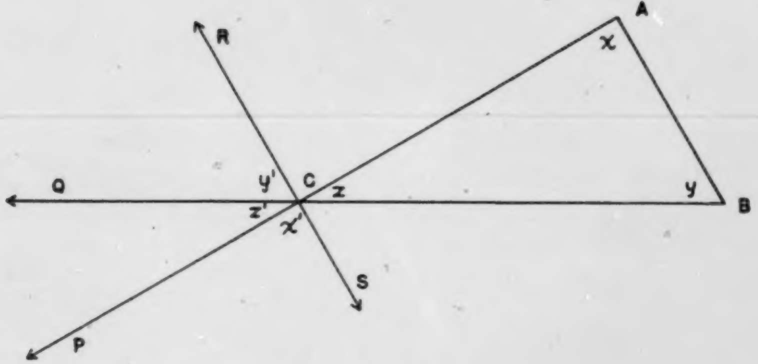
The following figures show another way of finding out about the sum of the angles of a triangle.



(a) (b)

(c) (d)

Consider the triangle ABC and the rays $\overrightarrow{AP}$ and $\overrightarrow{BQ}$ shown below. Line $\overleftrightarrow{RS}$ is drawn through point C so that the measure of $\angle y$ and the measure of $\angle y'$ are equal. Here a new notation, y' is used, y' is read "y prime." (In this problem this notation is used in naming angles.)



Use a property to explain "why" for each of the following whenever you can:

(a) Is $\overleftrightarrow{RS}$ parallel to $\overline{AB}$? Why?

(b) What kind of angles are the pair of angles marked x and x'? Is $m(\angle x) = m(\angle x')$? Why?

(c) What kind of angles are the pair of angles marked z and z'? Is $m(\angle z) = m(\angle z')$? Why?

(d) $m(\angle y) = m(\angle y')$ Why?

(e) $m(\angle x) + m(\angle y) + m(\angle z) = m(\angle x') + m(\angle y') + m(\angle z')$ Why?

(f) $m(\angle x) + m(\angle y) + m(\angle z)$ is the sum of the measures of the angles of the triangle. Why?

(g) $m(\angle x') + m(\angle y') + m(\angle z') = 180$ Why?

(h) $m(\angle x) + m(\angle y) + m(\angle z) = 180$ Why?

(i) Therefore the sum of the measures of the angles of the triangle is 180. Why?

This is a proof of Property 4.

Property 4: The sum of the measures in degrees of the angles of any triangle is 180.

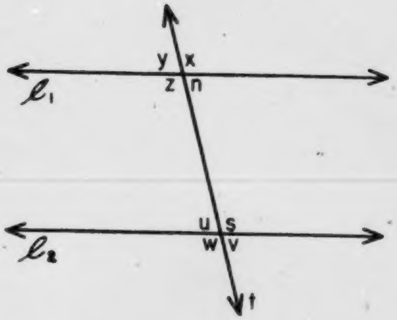
Notice that in this proof just any triangle was drawn. Does this proof apply to all triangles? If you are in doubt about this, you might draw some other triangles quite different in shape from the one in this section, label points, angles, segments, rays, and lines in the same way. Then, try the proof above for the figure you have drawn.

Exercises 3-5

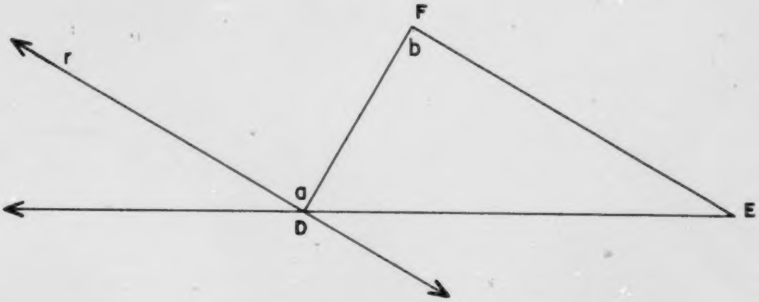
1. What is the measure of each angle of an equilateral triangle?
2. What is the measure of the third angle of the triangles if two of the angles of the triangle have the following measures?
   - (a) 40 and 80.
   - (b) 100 and 50.
   - (c) 70 and 105.
   - (d) 80 and 80.
3. Suppose one angle of an isosceles triangle has a measure of 50. Find the measures of the other two angles. Are two different sets of answers possible?
4. If two triangles, ABC and DEF, are drawn so that $m(\angle BAC) = m(\angle EDF) = m(\angle BCA) = m(\angle EFD)$, what will be true about angles ABC and DEF? Upon what property is your answer based?
5. In each of the following the measures of certain parts of the triangle ABC are given, those of the sides in inches and those of the angles in degrees. You are asked to find the measure of some other part. In each case, give your reason.

Given	Find	Property
(a) $m(\angle ABC) = 60$, $m(\angle BCA) = 40$.	$m(\angle CAB)$	?
(b) $m(\angle CAB) = 52$, $m(\angle BCA) = 37$.	$m(\angle ABC)$	?
(c) $m(\angle ABC) = 40$, $m(\overline{AB}) = 2$, and $m(\overline{AC}) = 2$.	$m(\angle ACB)$	?
(d) $m(\overline{AB}) = 3$, $m(\overline{AC}) = 3$, $m(\overline{BC}) = 3$.	$m(\angle BCA)$	?
(e) $m(\angle BAC) = 100$, $m(\angle BCA) = 40$, and $m(\overline{AB}) = 4$.	$m(\overline{AC})$	?

6. In the figure at the right ℓ_1 and ℓ_2 are parallel.
   (a) Does $m(\angle y) = m(\angle n)$? Why?
   (b) Does $m(\angle y) = m(\angle u)$? Why?
   (c) Try to prove that $m(\angle n) = m(\angle u)$.



7. In problem 6, it has been shown that the measures of angles a and b in the figure below are equal if line r is parallel to $\overleftrightarrow{EF}$. Use this property to prove that the sum of the measures of the angles of triangle DEF shown below is 180.



3-6. Parallelograms

In the following figure, k_1 and k_2 represent parallel lines. Lines a, b, and c are perpendicular to K_2 and go through three points: D, E, and F of K_1. That is, the lengths of $\overline{FA}$, $\overline{EB}$, and $\overline{DC}$ are the distances from F, E, and D to line k_2. One often draws a small square, as in the figure at A, B, C, to indicate that an angle is intended to be a right angle.

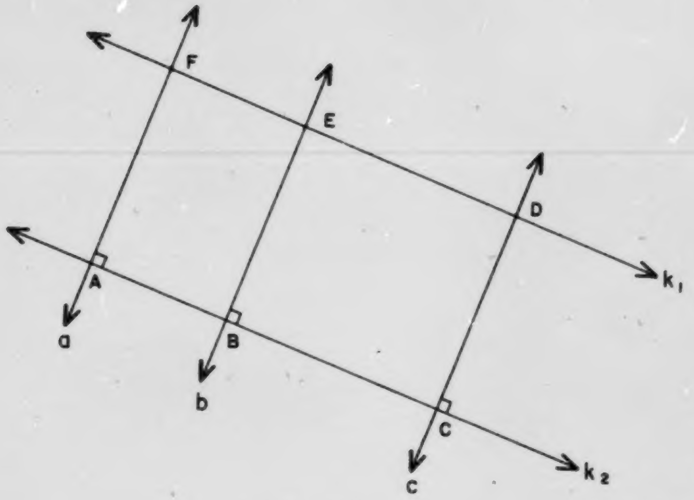


Figure 3-6a

In Figure 3-6a, $\overline{FA}$, $\overline{EB}$, and $\overline{DC}$ represent segments which are equal in length. This common length is called the distance between lines k_1 and k_2. Thus the distance between two parallel lines may be described as the length of any segment contained in a line perpendicular to the two lines, and having an endpoint on each of the lines.

Any simple closed curve which is a union of segments may be called a polygon. The word "polygon" is applied to curves which are not simple, but any polygon in this chapter will be a simple closed curve. Unless it is indicated otherwise, it is understood that a polygon lies in a plane.

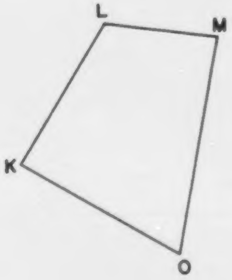
Polygons with different numbers of sides (i.e., segments) are given special names. A polygon with three sides is called a triangle. Similarly a polygon with four sides is called a quadrilateral, and a polygon with five sides is called a pentagon. In a quadrilateral two sides (segments) which do not intersect are called opposite sides.

A particularly important kind of quadrilateral is the parallelogram. This is a quadrilateral whose opposite sides lie on parallel lines. If two segments lie on parallel lines, the segments are said to be parallel. Thus, it is stated that the opposite sides of a parallelogram are parallel.

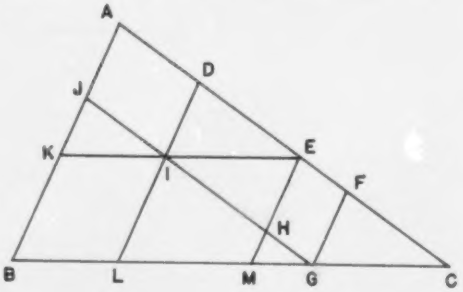
Property 5: Opposite sides of a parallelogram are parallel and congruent.

This property of congruence can be checked by measuring opposite sides of several parallelograms.

Exercises 3-6



1. The following questions refer to a figure which is a quadrilateral, as suggested by the drawing. Each question, however, involves a different quadrilateral.
   (a) $\overline{KL}$ is parallel to $\overline{OM}$, $\overline{LM}$ is parallel to $\overline{KO}$, $\overline{KL}$ has a length of 3 in. and $\overline{OK}$ a length of 6 in. What are the lengths of $\overline{LM}$ and $\overline{OM}$?
   (b) $\overleftrightarrow{KL} \cap \overleftrightarrow{OM}$ is the empty set, $\overleftrightarrow{LM} \cap \overleftrightarrow{OK}$ is the empty set. $\overline{LM}$ has a length of 4 in. and $\overline{OM}$ is three times as long as $\overline{LM}$. Find the lengths of $\overline{KL}$ and $\overline{OK}$.
   (c) $\overleftrightarrow{LM} \cap \overleftrightarrow{OK}$ is the empty set, $\overleftrightarrow{OM} \cap \overleftrightarrow{KL}$ is not the empty set. Can two opposite sides have the same length? Can both pairs of opposite sides have this property? (Draw figures to illustrate this.)
2. Draw a parallelogram and cut carefully along its sides. The resulting paper represents the interior of the parallelogram. Draw a diagonal (a line joining opposite vertices) and cut the paper along this diagonal. Compare the two triangular pieces. What do you conclude about these triangular pieces? Carry out the same process for two other parallelograms of different shapes. Write a statement that appears to be true on the basis of your experience in the problem.
3. In triangle ABC shown at the right, assume that the segments $\overline{AB}$, $\overline{DL}$, $\overline{EM}$, $\overline{FG}$ are parallel; assume that the segments $\overline{AC}$ and $\overline{JG}$ are parallel; and, assume that the segments $\overline{BC}$ and $\overline{KE}$ are parallel.



(a) List the parallelograms in the drawing. (There should be 10 parallelograms.)

(b) Without measuring, list the segments in the above figure which are congruent to $\overline{AJ}$.

(c) Without measuring, list the segments in the above figure which are congruent to $\overline{BK}$.

(d) Without measuring, list the segments in the above figure which are congruent to $\overline{GI}$.

3-7. Areas of Parallelograms and Triangles

If the sides of a parallelogram are extended by dotted lines, as shown in the Figure 3-7a,

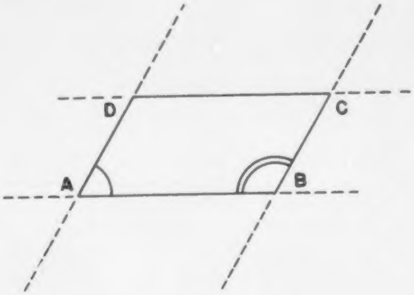


Figure 3-7a

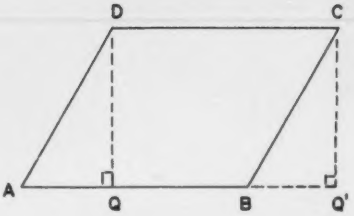
there are several pairs of parallel lines cut by a transversal.

By making use of the converse of Property 2a, one can verify Properties 6a and 6b about the angles of a parallelogram.

Property 6a: The angles of a parallelogram at two consecutive vertices are supplementary.

Property 6b: The angles of a parallelogram at two opposite vertices are congruent.

According to the results above, what can you conclude about the parallelogram if $\angle A$. is a right angle? If the figure is not a rectangle then $\angle A$ and $\angle B$ are not right angles, and one of them is an acute angle. Why? Suppose the acute angle is $\angle A$. From point D, draw the segment $\overline{DQ}$ perpendicular to the base $\overline{AB}$ of the parallelogram as shown in the figure at the

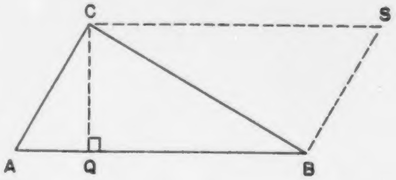


right. Since $\overline{AD}$ and $\overline{BC}$ are congruent, imagine the triangle AQD moved rigidly, that is, without changing its size and shape, into the position of triangle BQ'C. Then point Q' lies on the extension of $\overline{AB}$. Why?

The figure QQ'CD is therefore a rectangle. Moreover the rectangle QQ'CD and the parallelogram ABCD are made up of pieces of the same size. To find the area of the parallelogram it is only necessary therefore to find the area of the rectangle.

Notice that the base $\overline{AB}$ of the parallelogram is congruent to the side $\overline{QQ'}$ of the rectangle. $\overline{QD}$ is a segment perpendicular to the parallel lines $\overleftrightarrow{AB}$ and $\overleftrightarrow{CD}$ and is called an altitude of the parallelogram to the base $\overline{AB}$.

On the basis of the discussion above the following statement holds: "The number of square units of area in the parallelogram is the product of the number of linear units in the base and the number of linear units in the altitude to this base.



Consider any triangle ABC as shown at the right. Through C and B draw lines parallel to segments $\overline{AB}$ and $\overline{AC}$ and meeting in some point S. The figure ABSC is therefore a parallelogram. The segment $\overline{CQ}$ through C perpendicular to line $\overleftrightarrow{AB}$ is called the altitude of the triangle ABC to the base $\overline{AB}$. The length of altitude $\overline{CQ}$ is the distance from C to line $\overleftrightarrow{AB}$. Notice that $\overline{AB}$ and $\overline{CQ}$ are also a base and an altitude of the parallelogram.

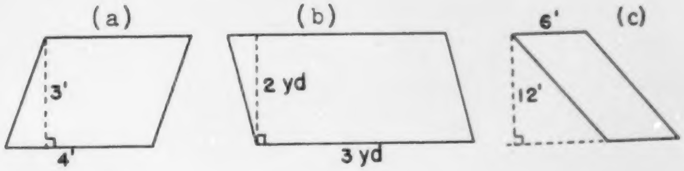
In Exercises 3-6, Problem 2, you discovered that the areas of triangles ABC and SCB are the same. Since the two triangular regions cover the whole parallelogram and its interior, it follows that the area of the triangle ABC is one half that of the area of the parallelogram ABSC. "The number of square units in the area of a triangle is one-half the product of the number of linear units in the base and the number of linear units in the altitude to this base."

Area of a parallelogram: If A = area, b the base, and h the altitude, then $A = bh$.

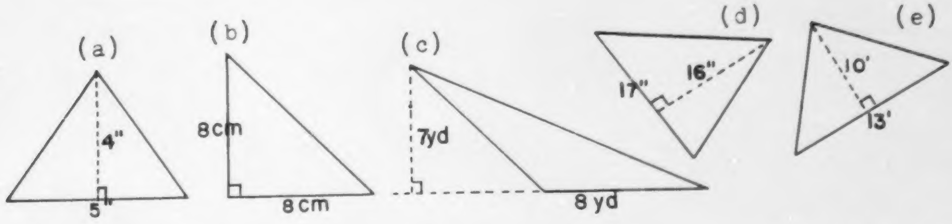
Area of a triangle: If A = area, b the base, and h the altitude, then $A = \frac{1}{2}bh$.

Exercises 3-7

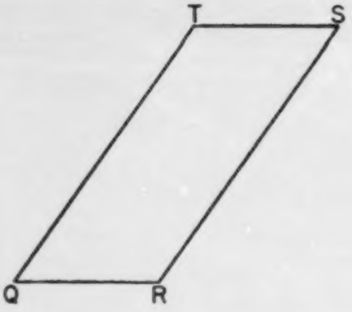
1. Find the areas of the parallelograms shown, using the dimensions given.

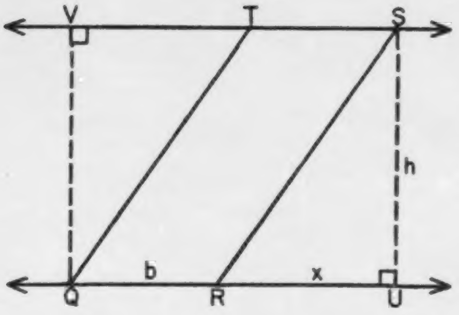


2. Find the areas of the triangles shown using the dimensions given.



3. Let QRST be any parallelogram not a rectangle. One possible drawing is shown below. Extend the line segments $\overline{TS}$ and $\overline{QR}$ as shown in the second figure below. At the vertices Q and S, where the angles of the parallelogram are acute, draw the perpendiculars $\overline{QV}$ and $\overline{SU}$. QUSV is a rectangle. Let b be the measure of $\overline{QR}$, h the measure of $\overline{US}$, and x the measure of $\overline{RU}$.





(a) If the measure of $\overline{QU}$ is $b + x$, what is the measure of $\overline{VS}$?

(b) What is the measure of $\overline{QV}$?

(c) What is the measure of $\overline{TS}$?

(d) What is the measure of $\overline{VT}$?

(e) What is the area of QUSV?

(f) What is the area of the triangle RUS?

(g) What is the area of the triangle QVT?

(h) Using your answers from (e), (f), and (g) above, show that the area of QRST is given by the sentence

$$A = bh.$$

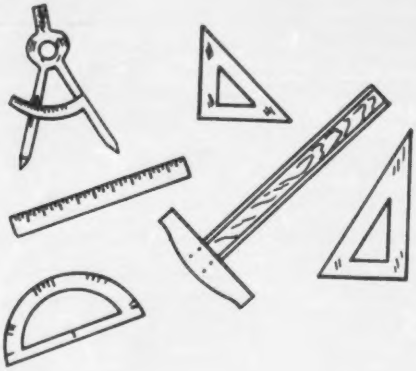
Chapter 4

CONSTRUCTIONS AND CONGRUENT TRIANGLES

4-1. Mathematical Drawings

Drawing pictures and diagrams helps us solve many problems. For some problems the drawings need not take long to make and need not be carefully drawn. For such problems a rough sketch of the situation is often sufficient. Other problems can be solved by measuring drawings, but when drawings are used this way they should be accurate representations.

Many tools are used to make accurate drawings. A man whose job is to make accurate drawings is called a draftsman. He uses a compass and a straightedge, but also he uses many other tools. Draftsmen use such tools as protractors, T-squares, 30-60 triangles, 45-45 triangles, rulers, parallel rulers, pantographs, and French Curves to help make drawings accurate. You will be using some of these tools but others are too expensive to obtain at this time.



Relationships among two or more lines on a plane are of special interest. Perpendicular lines are lines that intersect so as to form 90° angles. Parallel lines (in a plane) are two or more lines that do not intersect, or, in set language, lines whose intersection is the empty set. Perpendiculars may be drawn accurately with the aid of a protractor.

Protractors can also be used to draw parallel lines accurately if you remember that corresponding angles are equal when formed by parallel lines and a transversal. Suppose you wish to draw a line parallel to a given line. One way to do this is to draw a line perpendicular to the given line. Then any line perpendicular to it is parallel to the given line.

Triangles can be drawn with a ruler and protractor if you know the measure of two sides and the angle between those sides, or if you know the measures of two angles and the length of the side between the angles.

For the first set of measures, two sides and the angle between the sides, follow these steps:

a. Use your protractor to draw an angle of the given size.
b. Use your ruler to measure one side of the triangle on one ray of the angle. Be sure the measure starts at the vertex.
c. Measure the other side on the second ray of the angle.
d. Connect the endpoints of the segments to form the third side.

Scale drawings are representations of figures and are very important in many types of work. The first step in scale drawing is to select a scale by which the measurements of the object under construction may be drawn on your paper. Suppose you wish to draw a representation of a football field. A football field is a rectangle which is 100 yards long and $53\frac{1}{3}$ yards wide. If you use $\frac{1}{16}$ of an inch to represent one yard, the length of the field would be $6\frac{1}{4}$ inches on your paper. This number is obtained as follows:

$$\frac{\frac{1}{16}}{1} = \frac{x}{100},$$

$$x = 6\frac{1}{4}.$$

Let us now find the width.

$$\frac{\frac{1}{16}}{1} = \frac{x}{\frac{160}{3}},$$

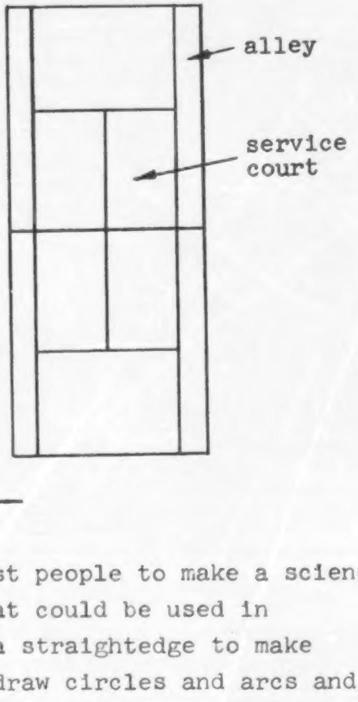
$$x = 3\frac{1}{3}.$$

Exercises 4-1

1. (a) Draw a horizontal line and mark two points, A and B, on it. Draw perpendiculars at each point.

(b) What is the relationship between the last two lines drawn?

2. Draw an oblique line and mark two points A and B. Draw lines through points A and B so that they are parallel to each other and so that each line intersects the oblique line forming angles of 60°.
3. (a) Draw a triangle that has two angles of 60° with a side $2\frac{1}{2}$ inches long between the angles.
   (b) What kind of triangle is this?
4. Draw a triangle with sides of $2\frac{1}{4}$ inches and $1\frac{7}{8}$ inches, with the angle formed by these sides being 110°.
5. City B is 40 miles east of city A and city C is 30 miles north of city B. Using $\frac{1}{8}$ inch to represent one mile draw a map in the form of a scale drawing of these distances. How many miles is it from city A to city C in a direct line?
6. Make a scale drawing of a tennis court. A tennis court is a rectangle having a length of 78 feet and a width of 36 feet. The alleys are $4\frac{1}{2}$ feet wide. The service courts are 21 feet in length and $13\frac{1}{2}$ feet in width. Let $\frac{1}{8}$ inch represent 1 foot.



4-2. Constructions

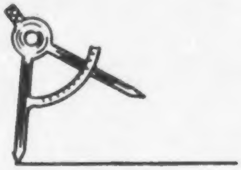
The ancient Greeks, who were the first people to make a science of geometry, had rules about the tools that could be used in geometry. Only two tools could be used; a straightedge to make straight line segments, and a compass to draw circles and arcs and to measure segments of equal length. An arc is any part of a circle. Notice that a compass will not measure a segment in inches or any other unit, but it can be used to construct a segment the same length as some other segment. In this chapter geometric drawings made with compass and straightedge are called constructions.

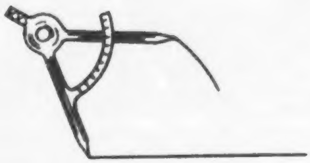
In the problems which occur at the end of this section, some terms are used with which you may not be familiar. When three or more lines intersect in one point, they are called concurrent lines.

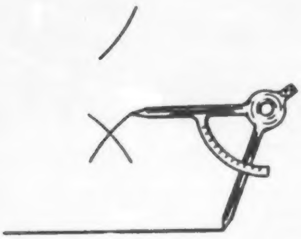
A polygon with four sides is called a quadrilateral; one with five sides is called a pentagon, with six sides a hexagon, and one with eight sides an octagon. If the sides are all the same length and the angles all have the same measure, it is called a regular polygon. A polygon is said to be "inscribed in a circle" if every vertex of the polygon is a point on the circle.

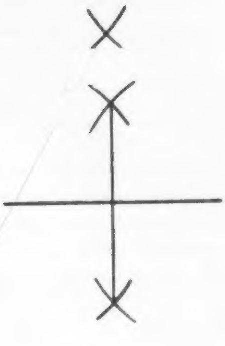
1. Bisecting a line segment

The figures show the steps in bisecting a line segment with straightedge and compasses.



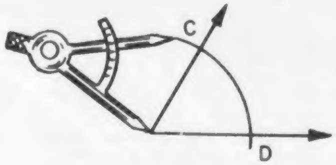


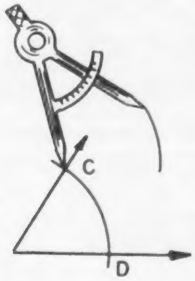


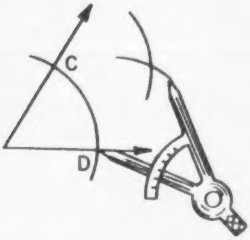


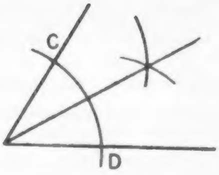
2. Bisecting an angle

The figures show the steps in bisecting an angle with straightedge and compasses.



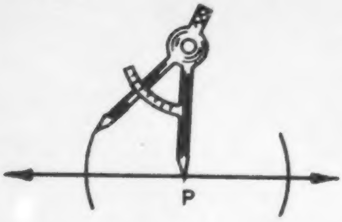


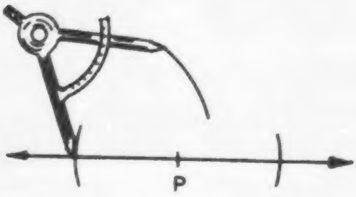


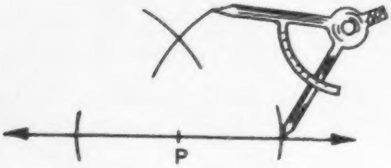


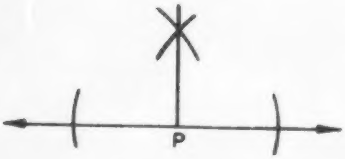
3. Erecting a perpendicular from a point on a line.

Follow the steps in the figures





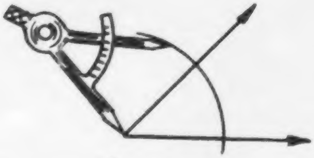


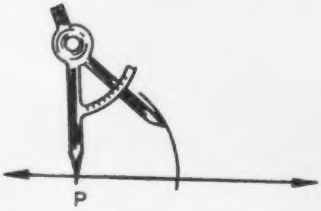


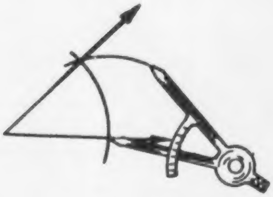
This construction can be modified to construct a perpendicular to a line from a point not on the line.

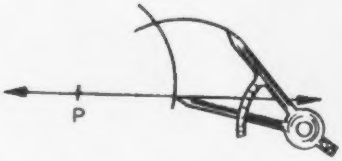
4. Copying an angle

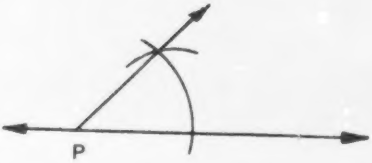
Follow the steps in the figures.





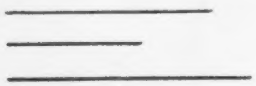




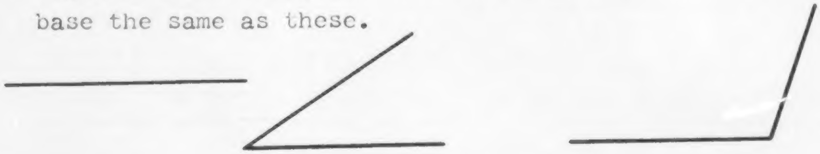


Exercises 4-2

1. Use a ruler to draw a horizontal line 1 $\frac{1}{2}$ inches long. Construct a vertical segment the same length.
2. Use a ruler to draw an oblique segment 3 $\frac{3}{8}$ inches long. Construct a horizontal segment the same length.
3. Bisect each segment that you constructed in problems 1-2. Use only compass and straightedge.
4. Draw an acute angle and bisect it. Use compass and straightedge.
5. Draw an obtuse angle and bisect it. Use compass and straightedge.
6. Draw a segment and then divide it into 4 equal parts. Use compass and straightedge only.
7. Draw an acute angle and an obtuse angle. Copy each angle using compass and straightedge only.
8. (a) Draw a triangle. Then construct the bisector of each angle. Extend the bisectors until they cross.
   (b) What do you notice about the figure?
9. Draw an obtuse angle and divide it into four equal angles. Use compass and straightedge.
10. (a) Draw a triangle. Then erect perpendiculars from each vertex to the opposite side. Extend the perpendiculars until they cross. (It may be necessary to extend the sides of the triangle so that the perpendicular meets this line.)
    (b) What do you notice in this figure?
11. Construct a triangle whose sides are the same length as the segments given here.

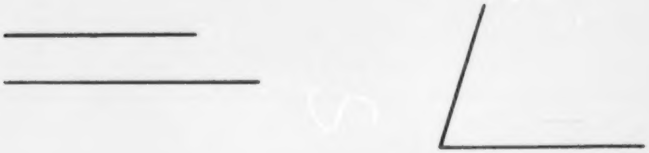


12. (a) Construct a triangle with a base the same length as the segment drawn here, and with the angles at each end of the base the same as these.



(b) Will all triangles constructed with these measures look alike? This construction is used to draw triangles when two angles and the side between these angles are known.

13. (a) Construct a triangle with two sides the same size as these segments and with the angle formed by these segments the same size as the angle drawn here.



(b) Will all triangles constructed with these measures look alike? This construction is used when two sides and the angle between those sides are known.

14. (a) Construct a right triangle that has one acute angle of 60°. Hint: How can an equilateral triangle be used as the basis for this? How many degrees in each angle of an equilateral triangle? How can you make two right triangles from an equilateral triangle?

(b) How many degrees are there in the measure of the third angle of this particular right triangle?

15. Draw three rays such that the endpoints of the rays are the only point of intersection.

16. How many angles are formed by the rays in problem 15?

17. Sets of concurrent lines that are related to triangles are:

The perpendicular bisectors of the sides,
The bisectors of the angles,
The perpendiculars from each vertex to the opposite side, (These perpendiculars are the altitudes of heights of the triangle.).

(a) Draw a triangle, then find, by construction, the midpoint of each side. Connect each of these midpoints to the opposite vertex. These segments are called the medians of a triangle.

(b) Are the medians concurrent?

18. Draw a circle. With the same radius, use any point on the circle as center and mark off arcs on the circle at equal distances from the point. Move the point of the compass to one point where the arc crosses the circle. Mark another arc on the circle. Continue until the arc drawn falls at the starting point. If you do this carefully you will discover that the last arc drawn falls exactly on the first point.
    a. How many arcs are there?
    b. Connect each intersection of the circle and an arc to the intersection on each side of it.
    c. What figure do these segments form?
    d. How can you use these points to construct an equilateral triangle?
    e. How can you form a six-pointed star?
    f. Using the same radius as the circle, draw an arc from one point on the circle to another. Move the point of the compass to either intersection and repeat. Continue around the circle. What does this figure look like?
19. Draw a circle and one diameter. Construct a diameter perpendicular to the first diameter. Connect the endpoints of the diameter in order.
    a. What figure does this form?
    b. How can you form a polygon with twice as many sides? There are two ways that this can be done. Can you find both of them?

4-3. Symmetry

This section is developed so that you will be able to discover for yourself what is meant by symmetry.

Exercises 4-3

1. (a) Fold a sheet of notebook paper down the middle. Starting at the folded edge, cut off a right triangle with the longer leg along the fold, as in Figure 4-3a. Unfold the piece you have cut off. What shape is it?

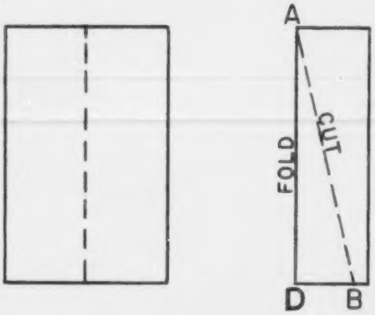


Figure 4-3a

(b) Label as A the vertex at the fold, and the other vertices as B and C. Label as D the intersection of the fold and the side. The figure now resembles Figure 4-3b.

(c) Refold your triangle along AD. Do right triangles A B D and A C D exactly fit over each other? It is said that triangle ABC has symmetry with respect to the line AD because when folded along AD the two halves exactly fit. Line AD is an axis of symmetry of the triangle.

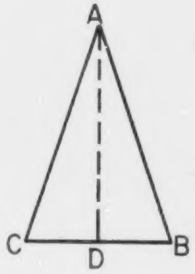


Figure 4-3b

2. (a) Take another piece of notebook paper and fold it lengthwise down the middle.

(b) Take your folded sheet and fold it crosswise down the middle (so that the crease folds along itself.) Cut off the corner where the folds meet, as indicated in Fig. 4-3c. Unfold the piece you cut off. What shape is it?

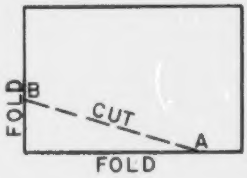


Figure 4-3c

(c) Label your figure as in Fig. 4-3d. If you fold along $\overline{AC}$ do the two halves exactly fit? What happens if you fold along $\overline{DB}$? Is there an axis of symmetry? How many?

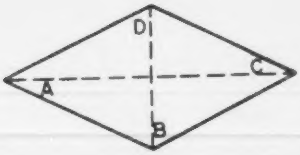


Figure 4-3d

3. Look at the regular hexagon in Figure 4-3e. Does each dotted line determine an axis of symmetry? There are other axes of symmetry. Find them. How many axes of symmetry does a regular hexagon have?

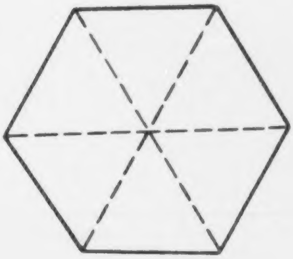


Figure 4-3e

4. Draw a circle and one of its diameters. Is this diameter an axis of symmetry. Does a circle have other axes of symmetry? Are there 5 axes of symmetry? 100? 10^5? Are there more than any number you may name?

5. Look at the ellipse in Fig. 4-3f. It is a figure you get if you slice off the tip of a cone but do not slice straight across. Is AB an axis of symmetry? Are there others? How many axes of symmetry does an ellipse have? $\overline{AB}$ is called the major axis of the ellipse. On another axis of symmetry is a segment called the minor axis of the ellipse. Why do you think $\overline{AB}$ is called the major axis? Where is the minor axis?

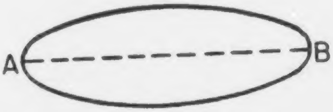


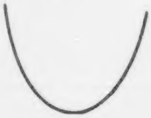
Figure 4-3f

From these exercises you have learned that many of the geometric figures you know are symmetrical with respect to a line. Many ornamental designs and decorations also have such symmetry.

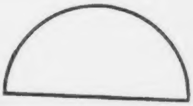
Definition: A figure is symmetrical with respect to a line if for each point A on the figure there is a point B on the figure for which the line is the perpendicular bisector of $\overline{AB}$.

6. Draw a rectangle and draw its axes of symmetry. Label each axis of symmetry. How many axes of symmetry does a rectangle have?
7. Draw an equilateral triangle, and label each axis of symmetry. How many axes of symmetry are there?
8. Draw a square, and label each axis of symmetry. How many axes of symmetry does a square have?
9. Draw and label the axes of symmetry, if there are any, for each of the figures. How many axes of symmetry does each figure have?

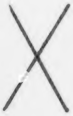
(a)



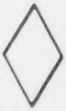
(b)



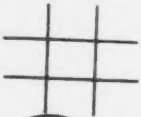
(c)



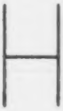
(d)



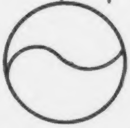
(e)



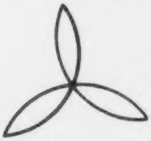
(f)



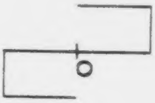
(g)



(h)



10. It is said that a circle is symmetrical with respect to a point, its center, and that an ellipse is symmetrical with respect to a point, its center (the point where its major and minor axes intersect). It is also said that the figure below is symmetrical with respect to point O. Describe in your own words what you think is meant by symmetry with respect to a point.



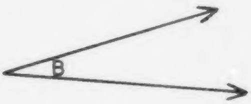
Which of the figures in Problem 9 have symmetry with respect to a point?

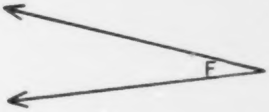
11. When an orange is cut through the center in such a way that each section is cut in half, you may think of the surfaces made by the cut as symmetrical. Symmetry of this kind is symmetry with respect to a plane. Name other objects that are symmetrical with respect to a plane.

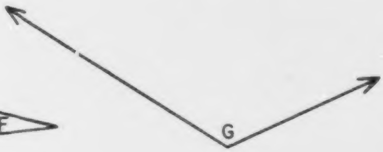
4-4. Congruent Triangles

In Section 4-3 when you cut along a right triangle on a folded sheet of paper, you produced an isosceles triangle. The axis of symmetry (the fold) forms, with the edges, two right triangles which have the same size and shape. When two figures have the same size and shape they are congruent. The two right triangles are congruent triangles.

Two circles, each with a radius of five inches are congruent, as are two line segments having the same length. Also, two rectangles are congruent if their bases and altitudes are congruent.







Angles B and F, as shown, have equal measures. You may say $\angle B$ is congruent to $\angle F$, and you may write $\angle B \cong \angle F$, where the symbol " $\cong$ " stands for the word "congruent". Is $\angle F \cong \angle G$?

Consider two congruent triangles, DEF and ABC traced on paper. If the paper is cut along the sides of triangle DEF, the paper model would represent a triangle and its interior. The paper model could be placed on triangle ABC and the two triangles would exactly fit. If point D were placed on point A with $\overline{DF}$ along $\overline{AC}$, point F would fall on point C, and point E would fall on point B. In these two triangles there would be these pairs of congruent segments and congruent angles:

$\overline{AB} \cong \overline{DE}$ $\qquad$ $\angle B \cong \angle E$

$\overline{CB} \cong \overline{FE}$ $\qquad$ $\angle C \cong \angle F$

$\overline{CA} \cong \overline{FD}$ $\qquad$ $\angle A \cong \angle D$

Recall that another way of expressing $\angle B \cong \angle E$ is $m(\angle B) = m(\angle E)$. Your choice of expression will depend upon whether you wish to emphasize the angles as being congruent figures or the measures as being equal numbers.

In triangle ABC, sides $\overline{AB}$ and $\overline{BC}$ and $\angle B$ are referred to as "two sides and the included angle". $\angle A$ and $\angle B$ and side $\overline{AB}$ are called "two angles with the included side". Do you see why?

In the study of constructions, we found that a triangle could be constructed if certain parts (sides and angles) are given. Three cases were considered:

(1) two sides and the included angle,
(2) two angles and the included side,
(3) three sides (provided that the measures of any one side is less than the sum of the measures of the other two sides).

You may wish to try these constructions again.

Furthermore, by construction, we see if the parts of a triangle are given, as in any one of the three cases, the triangle is completely determined. For any case, a correspondence can be established so that two triangles which have the given parts are congruent.

We state three properties:

Two triangles are congruent if two sides and the included angle of one triangle are congruent respectively to two sides and the included angle of the other triangle. This will be referred to as Property S. A. S. (Side, Angle, Side)

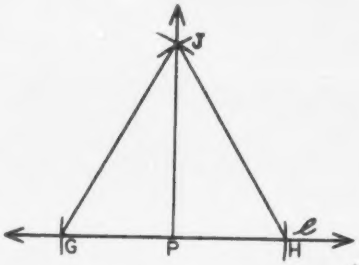
Two triangles are congruent if two angles and the included side of one triangle are congruent respectively to two angles and the included side of the other triangle. This will be referred to as Property A. S. A. (Angle, Side, Angle)

Two triangles are congruent if the three sides of one triangle are congruent respectively to the three sides of the other triangle. This will be referred to as Property S. S. S. (Side, Side, Side)

Let us look at the converse of the S.S.S. relationship. If two triangles are congruent then the three sides of one triangle are congruent respectively to the three sides of the other triangle. The converse of Property S.S.S. is a true statement. You can test this by looking again at constructions. Test the truth or falsity of the converse of Property S.A.S. and the converse of Property A.S.A. in the same manner.

Notice that the congruence sets up a one-to-one correspondence between pairs of sides of two congruent triangles, because you can let congruent sides correspond to each other. That is, suppose you call a, b, and c the sides of one triangle and r, s, and t the sides of a triangle congruent to it and suppose sides a and r, b and s, c and t are congruent. Then you may call a and r corresponding sides, b and s corresponding sides, and c and t corresponding sides. For this one-to-one correspondence it is true that if two triangles are congruent then their corresponding sides are congruent. You could set up the same kind of correspondence for angles and have: If two triangles are congruent then their corresponding angles are congruent.

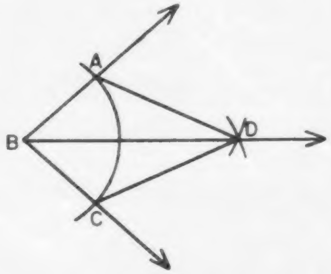
Recall the construction of a perpendicular to a line through a given point on the line. In this construction two arcs intersect at point J. If you draw the segments $\overline{GJ}$ and $\overline{JH}$, then there are formed two triangles, GPJ and HPJ. By construction $\overline{GP} \cong \overline{PH}$, $\overline{GJ} \cong \overline{HJ}$. The perpendicular to line ℓ has the segment $\overline{JP}$ on it. $\overline{JP}$ is a side of each triangle, and so is called a common side. By applying Property S.S.S., you know that triangle GPJ $\cong$ triangle PHJ, since three sides of one triangle are congruent to three sides of the other triangle. Angle JPG (opposite $\overline{GJ}$) in triangle GPJ corresponds to angle JPH (opposite $\overline{JH}$) in triangle PHJ. By applying the property that says if two triangles



are congruent then each pair of corresponding angles is congruent and each pair of corresponding sides is congruent, you know that $\angle$ JPG is congruent to $\angle$ JPH. If a protractor is laid along PH with the vertex mark at P and 0° mark on PH, then the 180° mark will be on the ray PG. This means that the sum of the measures of the angles at P is 180 and, since these measures are equal, each must be 90. Hence, angles JPG and JPH are right angles.

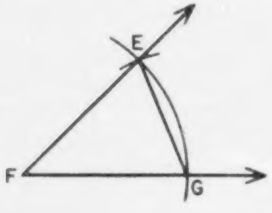
Exercises 4-4

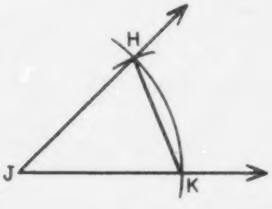
1. In the figure the construction of the bisector of $\angle$ ABC is shown. Two segments, $\overline{AD}$ and $\overline{CD}$ are drawn.



   (a) What parts of triangle ABD are congruent to corresponding parts of triangle CBD by construction?
   (b) Find another part of triangle ABD that must be congruent to a part of triangle BCD. Why are they congruent?
   (c) Is triangle ABD congruent to triangle BCD? Why?
   (d) Is $\angle$ ABD congruent to $\angle$ DBC? Why?

2. In the figures shown here the construction of $\angle$ HJK makes $\angle$ HJK $\cong$ $\angle$ EFG. Segments $\overline{EG}$ and $\overline{HK}$ are drawn.



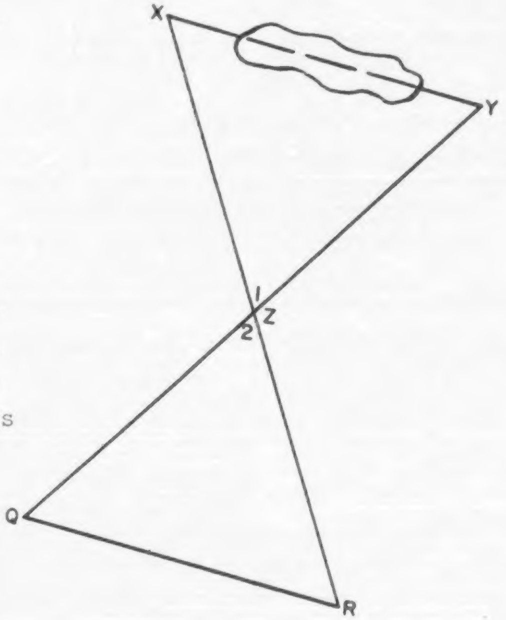


(a) What parts of triangle EFG are congruent to corresponding parts of triangle HJK by construction?

(b) Is triangle EFG congruent to triangle HJK? Why?

(c) Is $\angle J$ congruent to $\angle F$? Why?

3. Mr. Thompson wishes to measure the distance between two posts on edges of his property. A grove of trees between the two posts (X and Y) makes it impossible to measure the distance XY directly. He locates point Z such that he can lay out a line from X to Z and continue it as far as needed. Point Z is also in a position such that Mr. Thompson can lay out a line YZ and continue it as far as needed. Mr. Thompson knows that $\angle 1 \cong \angle 2$ since they are vertical angles. He extends $\overline{YZ}$ so that $\overline{QZ} \cong \overline{YZ}$ and $\overline{XZ}$ so that $\overline{XZ} \cong \overline{ZR}$.

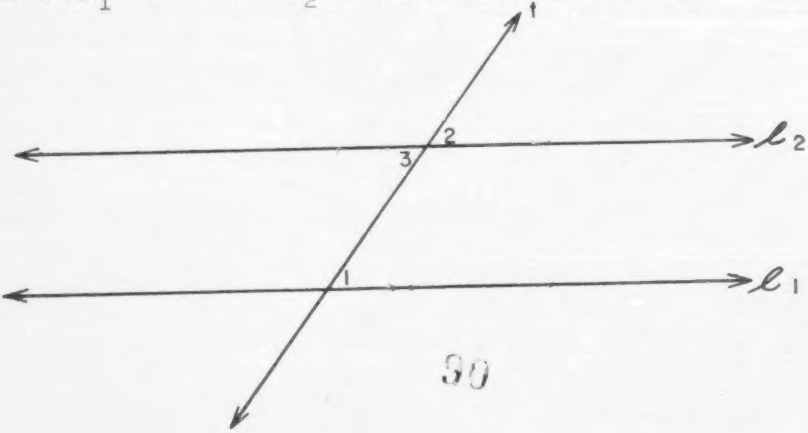


(a) How is point R located? Is $\overline{XZ} \cong \overline{ZR}$?

(b) Are triangles XYZ and QZR congruent? Why?

(c) How can Mr. Thompson determine the length of $\overline{XY}$?

4. Line ℓ_1 and line ℓ_2 are parallel lines cut by a transversal.

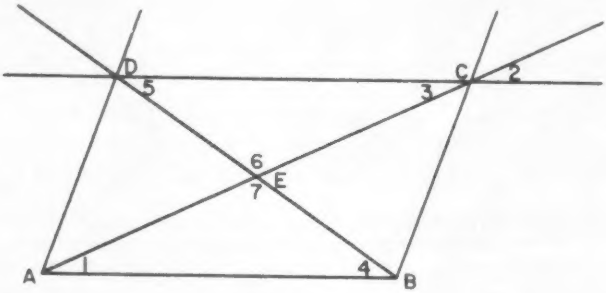


(a) What do you know about angles 1 and 2?

(b) Are angles 2 and 3 congruent? Why?

(c) Show that $\angle 1 \cong \angle 3$.

5. In the parallelogram, ABCD, the diagonals $\overline{AC}$ and $\overline{BD}$ intersect at E.



(a) Is angle 1 (in $\triangle ABE$) congruent to angle 2?

(b) What kind of angles are $\angle 2$ and $\angle 3$? Are they congruent?

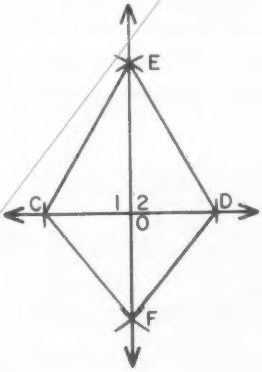
(c) How does the size of $\angle 1$ compare with the size of $\angle 3$?

(d) Show that $\angle 6 \cong \angle 7$ and that $\angle 5 \cong \angle 4$.

(e) When two triangles have three pairs of congruent angles, are the triangles always congruent? If not, what else is needed?

(f) Is any side of $\triangle ABE$ congruent to the corresponding side of $\triangle CDE$?

(g) Show that the diagonals of a parallelogram bisect each other.



6. The construction of the perpendicular bisector of segment $\overline{CD}$ is shown. Usually the same radius is used for the four arcs. However, it is only necessary for the two arcs that intersect on one side of the segment to have the equal radii. Thus, the arcs drawn from C and D that intersect at E have equal radii, and the two arcs

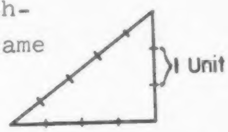
drawn from C and D that intersect at F have equal radii. By applying some of the properties about congruent triangles, show why $\overline{EF}$ bisects $\overline{CD}$ and is perpendicular to $\overline{CD}$. Hint: First think about the large triangles CFE and DFE, then about another pair of triangles that seem to be congruent.

4-5. Right Triangles

If a triangle contains an angle having a measure, in degrees, 90, it is called a *right triangle*. If a triangle contains an obtuse angle (an angle with measure greater than 90), it is called an *obtuse triangle*. If a triangle contains only acute angles (with measures less than 90), it is called an *acute triangle*.

The right triangle is of special interest. The ancient Egyptians are said to have used a particular right triangle to make corners "square". This triangle has sides 3, 4, and 5 units long. When such a triangle is made of rope stretched taut, the angle between the two shorter sides is a right angle.

While the Egyptians are thought to have made use of this fact, it was left to the Greeks to prove the relationship involved. The Greek philosopher and mathematician, Pythagoras, who lived about 500 B.C. became interested in the problem. Pythagoras is credited with the proof of the basic property that will be studied in this section; this property is still known by his name, the Pythagorean Property.



It is thought that Pythagoras looked at a mosaic like the one pictured in Figure 4-5a. He noticed that there are many triangles of different sizes that can be found in the mosaic. But he noticed more than this. If each side of any triangle is used as one side of a square, the sum of the areas of the two smaller squares is the same as the area of the larger square. In Figure 4-5b, two triangles of different size are inked in and the squares drawn on the sides of the triangles shaded. Count the number of the smallest triangles in each square. For each triangle that is inked in, how does the number of small triangles in the two smaller squares compare with the number in the larger square? If you draw a mosaic like this, you will find that this is true not only for the two triangles given here but for a triangle of any size in this mosaic.

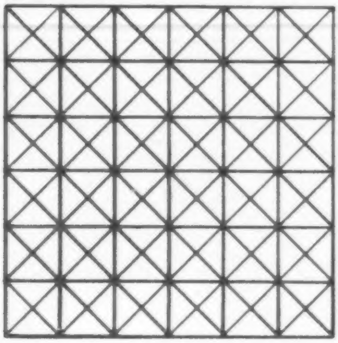


Figure 4-5a

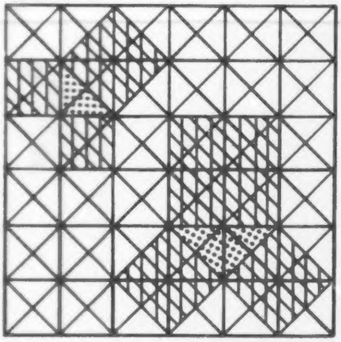
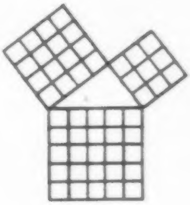


Figure 4-5b

Pythagoras probably noticed the same relation in the 3-4-5 triangle that the Egyptians had used for so long to make a right angle. The small square are each 1 square unit in size. In the three squares there are 9, 16 and 25 small squares. Notice that 9 + 16 = 25. Pythagoras was able to prove that in any right triangle, the area of the square on the hypotenuse (longest side) is equal to the sum of the areas of the squares on the other two sides. This is the Pythagorean Theorem, or as it will be called, the Pythagorean Property.



So far this has been shown only for two very special right triangles. It is true for all right triangles. In mathematical language the Pythagorean Property is $c^2 = a^2 + b^2$ where c stands for the measure of the hypotenuse and a and b stand for the measures of the other two sides. The measures of any two sides can be substituted in the number sentence above, and from this the third value can be found. A familiar triangle can be used to show this. If the two short sides are 3 units and 4 units, what is the square of the hypotenuse?

$$c^2 = a^2 + b^2$$

$$c^2 = 3^2 + 4^2$$

$$c^2 = 25.$$

$$c = 5.$$

Of course, 5×5 is 25, so $c = 5$; 5 is the positive square root of 25. If a number is the product of two equal factors, then each factor is a square root of the number. The symbol for the positive square root is $\sqrt{\ }$. The numeral is placed under the sign, for example, $\sqrt{25} = 5$.

What is $\sqrt{9}$? $\sqrt{16}$? $\sqrt{36}$? $\sqrt{30}$? The first three are easy to understand since $3 \times 3 = 9$, $4 \times 4 = 16$ and $6 \times 6 = 36$ but there is no integer that can be multiplied by itself to give the product 30. In fact, there is no rational number whose square is 30! There are decimal forms of rational numbers that give products close to 30 when squared. You can even find a number whose square is as close to 30 as you wish! For use now there is a table at the end of the chapter that gives the decimal value (to the nearest thousandth) that is closest to the square root of integers from 1 to 100. You can also use the table to find the square root of all counting numbers up to 10,000 that have rational square roots.

Exercises 4-5

When approximate values are used in these problems, use the symbol, $\approx$, in the work and answer.

1. Use the table at the end of the chapter to find the approximate value of:

   (a) $\sqrt{5}$ (b) $\sqrt{41}$ (c) $\sqrt{13}$

2. Use the Pythagorean Property to find the length of the hypotenuse for each of these triangles.

   (a) Length of side a is 1", length of b is 2"

   (b) Length of a is 4', length of b is 5'

(c) Length of a is 2", length of b is 3"

(d) Length of a is 5 yd and the length of b is 6 yd.

3. The hypotenuse of a right triangle is 13 ft. and one side is 5 ft. Find the length of the third side.

$$c^2 = a^2 + b^2$$
$$13^2 = 5^2 + b^2$$
$$169 - 25 = b^2$$
$$144 = b^2$$
$$12 = b$$

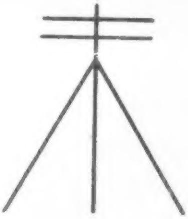
The third side is 12 feet long. Find the third side of these right triangles. The measurements are in feet.

(a) $c = 15$, $b = 9$

(b) $c = 26$, $a = 24$

(c) $c = 39$, $b = 15$

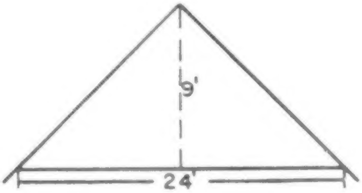
4. A telephone pole is steadied by guy wires as shown. Each wire is to be fastened 15 ft. above the ground and anchored 8 ft. from the base of the pole.



(a) How much wire is needed to stretch one wire from the ground to the point on the pole at which the wire is fastened?

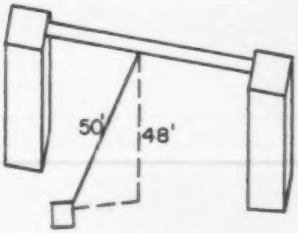
(b) If 5 ft. of wire are required to fasten each wire to the pole and the ground anchor, how much wire is needed for each pole?

5. A roof on a house is built as shown. How long should each rafter be if it extends 18 inches over the wall of the house?

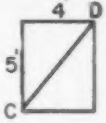


6. A hotel builds an addition across the street from the original building. A passageway is built between the two parts at the third floor level. The beams that support this

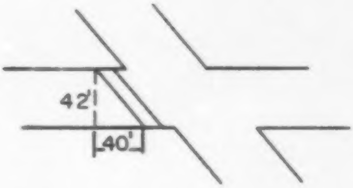
passage are 48 ft. above the street. A crane operator is lifting these beams into place with a crane arm that is 50 ft. long. How far down the street from a point directly under the beam should the crane cab be?



7. A garden gate is 4 ft. wide and 5 ft. high. How long should the brace that extends from C to D be?



8. Two streets meet at the angle shown. The streets are 42 ft. wide. Lines for a cross walk are painted so that the walk runs in the same direction as the street. If it is 40 ft. from one end of the cross walk to the point that is on the perpendicular from the other end point of the walk, how long is the crosswalk?



9. If you know that one side of an equilateral triangle is R units, what is its height? Give the answer in relation to R.

4-6. One Proof of the Pythagorean Property

There are many proofs of this property. The one used here is not the one used by Pythagoras.

Draw two squares the same size. Separate the first square into two squares and two rectangles as shown here:

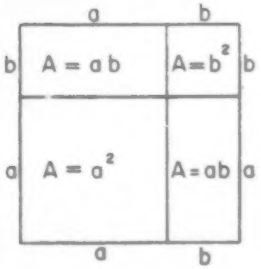


Figure 4-6a

Let the measure of each side of the larger square in Figure 4-6a be a and the measure of each side of the small square be b. Notice the areas of the small squares and rectangles.

One square has an area of measure a^2.

The other square has an area of measure b^2.

Each rectangle has an area of measure ab.

Since the area of Figure 4-6a is equal to the sum of the areas of all of its parts, the measure of the area of Figure 4-6a is

$$a^2 + 2\,(ab) + b^2$$

Now turn to the second square. Use the same numbers, a and b, that were used in the first square.

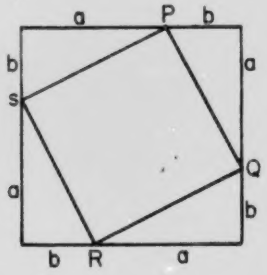


Figure 4-6b

Mark the lengths off as shown here and draw the segments $\overline{PQ}$, $\overline{QR}$, $\overline{RS}$ and $\overline{SP}$. The large square is separated into 4 triangles and a quadrilateral that appears to be a square.

The measure of each triangular area is $\frac{1}{2}\,ab$. There are four congruent triangles. The sum of the measures of the areas of all four triangles is $4(\frac{1}{2}\,ab)$ or $2ab$.

If you look back to Figure 4-6a, you will see that $2\,ab$ is the measure of the area of the two rectangles. Cut the two rectangles from the first square. Cut along the diagonal of each rectangle. See if the four triangles you cut are congruent with those in the second square.

$$A_{square} = a^2 + 2ab + b^2 \quad \text{(From Figure 4-6a)}$$

$$A_{square} = 4(\tfrac{1}{2}\,ab) + A_{PQRS} \quad \text{(From Figure 4-6b)}$$

$$= 2ab + A_{PQRS}$$

Therefore $a^2 + 2ab + b^2 = 2ab + A_{PQRS}$ Why?

$a^2 + b^2 = A_{PQRS}$ (addition property of equality.)

This shows that PQRS has an area whose measure is $a^2 + b^2$ units, but a^2 is the measure of the area of one small square in the first figure and b^2 is the measure of the area of the other square. From this the area of the figure in the center of the second square is equal to the sum of the areas of the two small squares.

Place the square whose area measure is a^2 along the side of length a of one triangle in the second square. Place the square whose area measure is b^2 along the side of length b of the same triangle. The areas of the squares on the two sides of the triangle are equal to the area of the figure in the center of Figure 4-6b. All you need to do now is prove that this figure is a square!

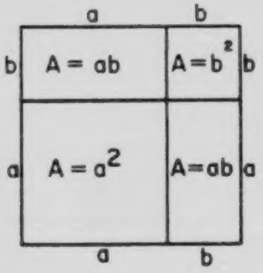
What are the properties of a square?

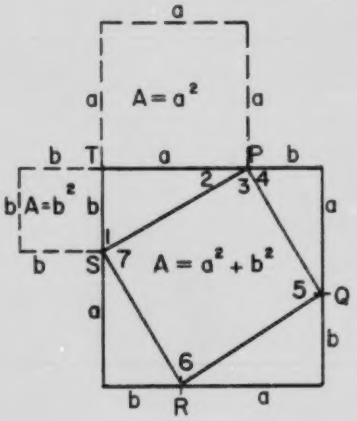
1. The four sides are congruent.
2. Each angle is 90° in measurement.

If you can prove that these two conditions for the quadrilateral in Figure 4-6b, the Pythagorean Property has been proved.

As has been stated, the four triangles are congruent since for each pair two corresponding sides and the angle between these sides are congruent. As a result, PQ = QR = RS = SQ because they are measures of corresponding segments of congruent triangles.

So far it has been shown that the squares in Figure 4-6a are congruent to the squares on the short sides of any one of the triangles in Figure 4-6b. It has also been shown that the sum of the areas of these squares is equal to the area of PQRS, and that PQRS has four congruent sides. Let us prove that the angles are right angles.





(1) In Δ PST, m($\angle$ 1) + m($\angle$ 2) = 90 Why?

(2) m($\angle$ 1) = m($\angle$ 4) Why?

(3) Therefore m($\angle$ 4) + m($\angle$ 2) = 90 Why?

(4) and m($\angle$ 2) + m ($\angle$ 3) + m($\angle$ 4) = 180 Why?

(5) m($\angle$ 3) + 90 = 180 Why?

(6) and m($\angle$ 3) = 90 Why?

You can go through the same type of reasoning to show that angles 5, 6 and 7 are also right angles.

PQRS has been proved to be a square and its area has also been proved equal to the sum of the areas of the squares on the other two sides.

4-7. Solids

You have been drawing figures which are contained within a plane. It is not so easy to draw pictures of solids on paper or on the chalkboard. This is because you must draw the figure on a surface in such a way that it will appear to have depth. This requires the use of projection which you have possibly studied in art.

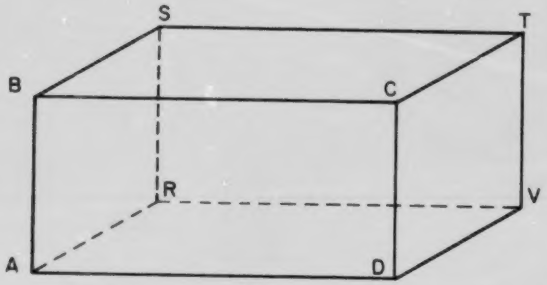
(a) Prisms

(1) Rectangular Prisms. One way to draw a box is as follows:

a. Draw a rectangle such as ABCD in the figure below.

b. Now draw a second rectangle RSTV in a position similar to the one in the figure.

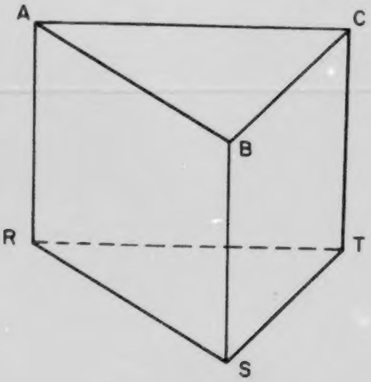
c. Draw $\overline{AR}$, $\overline{BS}$, $\overline{DV}$ and $\overline{CT}$.



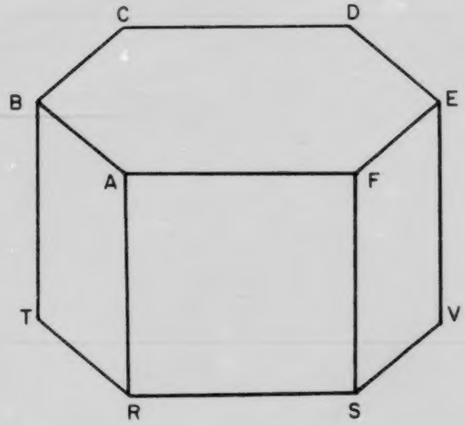
When you look at a solid you cannot see all of the edges, or faces, unless the solid is transparent. For this reason the edges, which are not visible, are represented by dotted line segments. This also helps to give the proper projection to the drawing. The dotted line segments do not have to be drawn.

After you have sketched a rectangular prism, you should be able to sketch other prisms by studying the sketches given.

(2) Triangular Prism



(3) Hexagonal Prism



(4) Pyramids

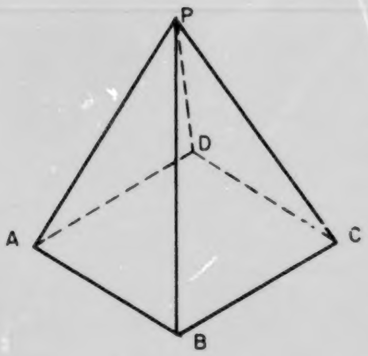
a. In this drawing let the base represent a square with a vertex at the bottom of the drawing.

b. First, draw only two sides of the square, such as $\overline{AB}$ and $\overline{BC}$, as shown in the figure below.

c. Now select a point P, directly above point B, and draw $\overline{PA}$, $\overline{PB}$ and $\overline{PC}$.

d. $\overline{AD}$, $\overline{CD}$ and $\overline{PD}$ may now be drawn as dotted line segments intersecting on $\overline{PB}$, with $\overline{AD}$ parallel to $\overline{BC}$ and $\overline{CD}$ parallel to $\overline{AB}$.

e. How many faces does this pyramid have?

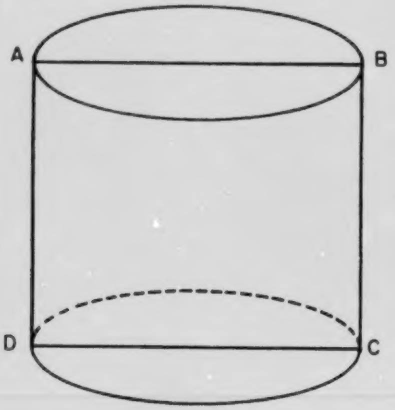


(5) Cylinders

A cylinder has two congruent bases which are circular regions. In drawing cylinders on the surface of your paper you are again confronted with the problem of getting the correct perspective. If you construct two circles with your compass and then draw line segments connecting the end points of parallel diameters of the two circles, the figure you have drawn will not have the appearance of a right cylinder. In order to have the proper perspective you must draw the bases as ovals as shown in the figure below.

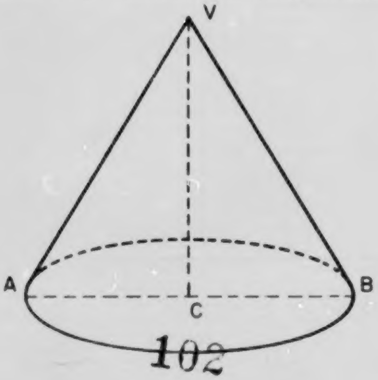
a. Draw a rectangle such as ABCD in the figure below.

b. Now use $\overline{AB}$ and $\overline{DC}$ as diameters of the circular bases to be shown.



(6) Cone

A cone may be sketched as shown.



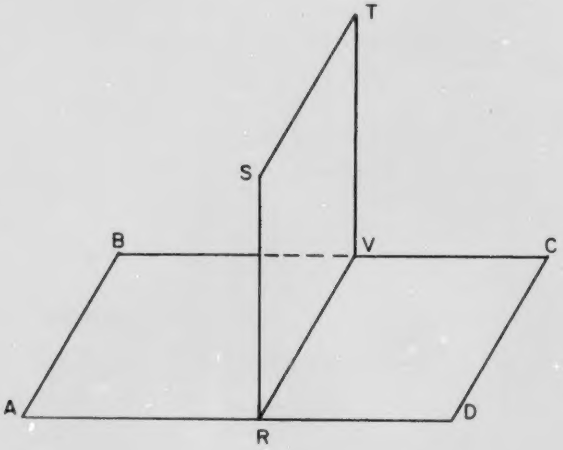
(7) Intersecting Planes

There are times when it is useful to represent, on a surface, the intersection of two or more planes. An example is the intersection of a wall and the floor of your room.

a. Draw a parallelogram ABCD.

b. Select a point R on $\overline{AD}$ and draw $\overline{RV}$ parallel to $\overline{AB}$.

c. Draw a perpendicular $\overline{VT}$ to $\overline{BC}$ and a perpendicular $\overline{RS}$ to $\overline{AD}$ so that $\overline{VT}$ and $\overline{RS}$ have equal measure. Now draw $\overline{TS}$.

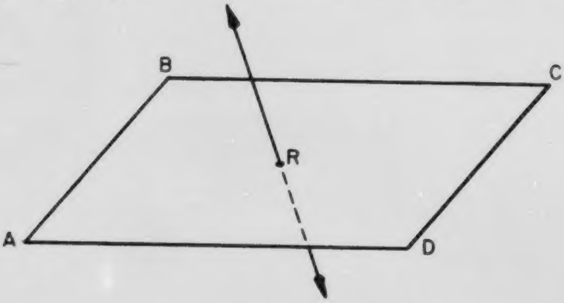


(8) Line Intersecting a Plane

This type of drawing is also useful at times. It is illustrated below.

a. Draw a parallelogram such as ABCD.

b. Select a point R on the surface of ABCD.



c. Now draw a line through R so that it appears to pass through the surface of ABCD. This will require some practice.

d. You will have a better picture if the line through R is not parallel to a side of the parallelogram.

Practice making sketches of each of the solids.

TABLE
SQUARES AND SQUARE ROOTS OF NUMBERS

No.	Squares	Square roots	No.	Squares	Square roots
1	1	1.000	36	1,296	6.000
2	4	1.414	37	1,369	6.083
3	9	1.732	38	1,444	6.164
4	16	2.000	39	1,521	6.245
5	25	2.236	40	1,600	6.325
6	36	2.449	41	1,681	6.403
7	49	2.646	42	1,764	6.481
8	64	2.828	43	1,849	6.557
9	81	3.000	44	1,936	6.633
10	100	3.162	45	2,025	6.708
11	121	3.317	46	2,116	6.782
12	144	3.464	47	2,209	6.856
13	169	3.606	48	2,304	6.928
14	196	3.742	49	2,401	7.000
15	225	3.873	50	2,500	7.071
16	256	4.000	51	2,601	7.141
17	289	4.123	52	2,704	7.211
18	324	4.243	53	2,809	7.280
19	361	4.359	54	2,916	7.348
20	400	4.472	55	3,025	7.416
21	441	4.583	56	3,136	7.483
22	484	4.690	57	3,249	7.550
23	529	4.796	58	3,364	7.616
24	576	4.899	59	3,481	7.681
25	625	5.000	60	3,600	7.746
26	676	5.099	61	3,721	7.810
27	729	5.196	62	3,844	7.874
28	784	5.292	63	3,969	7.937
29	841	5.385	64	4,096	8.000
30	900	5.477	65	4,225	8.062
31	961	5.568	66	4,356	8.124
32	1,024	5.657	67	4,489	8.185
33	1,089	5.745	68	4,624	8.246
34	1,156	5,831	69	4,761	8.307
35	1,225	5,916	70	4,900	8.367

No.	Squares	Square roots
71	5,041	8,426
72	5,184	8.485
73	5,329	8.544
74	5,476	8.602
75	5,625	8.660
76	5,776	8.718
77	5,929	8.775
78	6,084	8.832
79	6,241	8.888
80	6,400	8.944
81	6,561	9.000
82	6,724	9.055
83	6,889	9.110
84	7,056	9.165
85	7,225	9.220
86	7,396	9.274
87	7,569	9.327
88	7,744	9.381
89	7,921	9.434
90	8,100	9.487
91	8,281	9.539
92	8,464	9.592
93	8,649	9.644
94	8,836	9.695
95	9,025	9.747
96	9,216	9.798
97	9,409	9.849
98	9,604	9.899
99	9,801	9.950
100	10,000	10.000

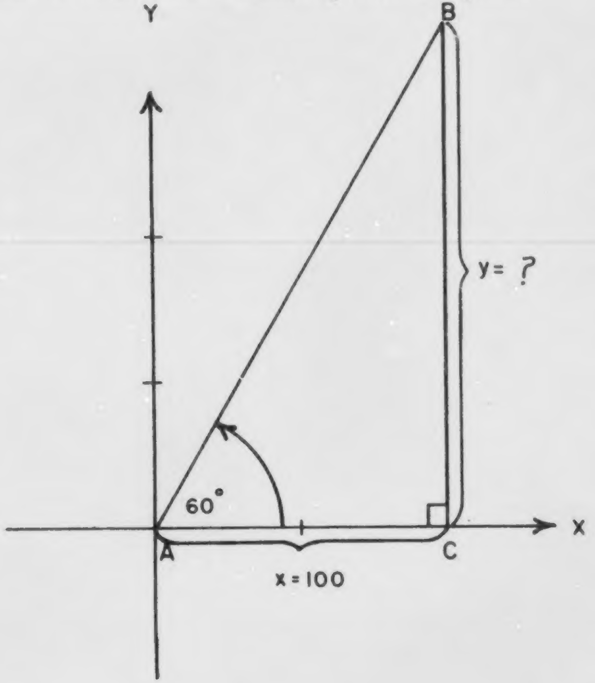
Chapter 5

SIMILAR TRIANGLES AND VARIATION

5-1. Indirect Measurement

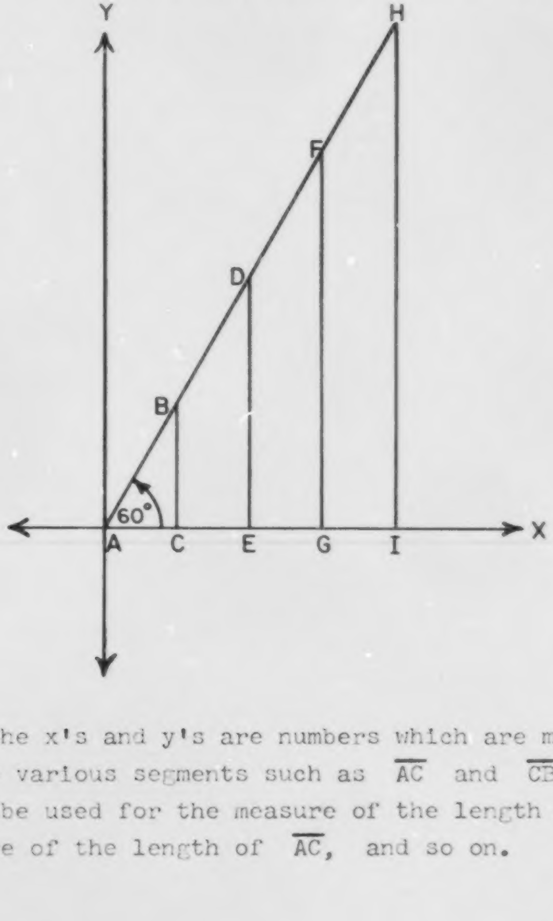
You may have read that the sun is 93,000,000 miles away from the earth, or that the distance from the earth to the nearest star (other than the sun) is 4 light-years, or that the diameter of the earth is about 8,000 miles. These distances are measured indirectly.

Indirect measurements are also used in problems which are closer to home. To measure the height of a building, you may sight the top of the building with a sextant and measure the angle of elevation from the ground. Then you measure the distance from where you sighted the top to the bottom of the building. In the following diagram it is indicated that the measurement of the angle of elevation is 60° and that the measure x of the distance from A to C, in feet, is 100. From this information the measure y of the height of the building in feet can be calculated. The x and y are used to represent numbers, not lengths.



Notice that in the above diagram a drawing has been made to scale. The triangle above is not of the same size as the real triangle, but it does have the same shape. It is said that this one and the actual triangle are similar. Measure the length of $\overline{CB}$ in the figure above. Calculate the height y in feet, given that 1 inch represents 50 feet.

In the diagram below four triangles have been drawn, each having one right angle and one 60° angle. They are $\triangle ABC$, $\triangle ADE$, $\triangle AFG$, $\triangle AHI$. What do you notice about their shapes? For each one find the measure x of the horizontal distance and y of the vertical distance, and calculate the ratio $\frac{y}{x}$.



In the table the x's and y's are numbers which are measures of the lengths of the various segments such as $\overline{AC}$ and $\overline{CB}$. The notation CB will also be used for the measure of the length of $\overline{CB}$, AC for the measure of the length of $\overline{AC}$, and so on.

Exercises 5-1a

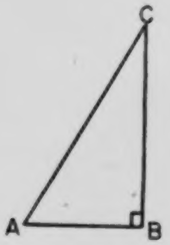
1. Complete the following table by measuring:

Right tri-angle					
ABC	$y_1 = CB =$	$x_1 = AC =$	$\frac{y_1}{x_1} = \frac{CB}{AC} =$	$\frac{x_1}{r_1} = \frac{AC}{AB} =$	$\frac{y_1}{r_1} = \frac{CB}{AB} =$
ADE	$y_2 = ED =$	$x_2 = AE =$	$\frac{y_2}{x_2} = \frac{ED}{AE} =$	$\frac{x_2}{r_2} = \frac{AE}{AD} =$	$\frac{y_2}{r_2} = \frac{ED}{AD} =$
AFG	$y_3 = GF =$	$x_3 = AG =$	$\frac{y_3}{x_3} = \frac{GF}{AG} =$	$\frac{x_3}{r_3} = \frac{AG}{AF} =$	$\frac{y_3}{r_3} = \frac{GF}{AF} =$
AHI	$y_4 = IH =$	$x_4 = AI =$	$\frac{y_4}{x_4} = \frac{IH}{AI} =$	$\frac{x_4}{r_4} = \frac{AI}{AH} =$	$\frac{y_4}{r_4} = \frac{IH}{AH} =$

As you found in completing the table above, for triangles with one right angle and one 60° angle, the ratios $\frac{y}{x}$ are the same for all x and y. Similarly $\frac{y}{r}$ and $\frac{x}{r}$ are the same for all y and r and all x and r. The four triangles have the same shape. They are similar triangles.

It appears that in similar triangles the corresponding sides are proportional. The ratio $\frac{y}{x}$ is a certain fixed number. If you know what this number is, then you can solve all problems of the following kind:

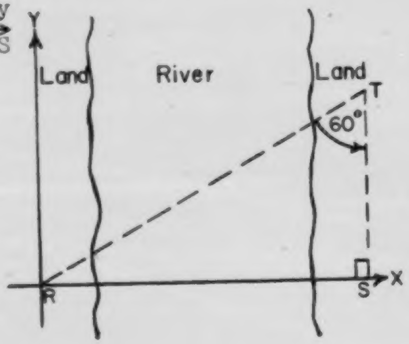
Let ABC be a triangle in which $\angle ABC$ is a 90° angle and $\angle BAC$ is a 60° angle. Given the length of the segment $\overline{AB}$, find the distance from B to C, or the length of $\overline{BC}$.



In general, in the right triangle ABC, with the right angle at B, the ratio $\frac{y}{x}$, of the measures $y = BC$ and $x = AC$, depends only on the measure of the angle BAC. Tables have been made, showing the value of such ratios as $\frac{y}{x}$ for different angles. You will learn how to use such tables to solve problems of indirect measurement.

Exercises 5-1b

1. Draw a right triangle ABC. Let D be the midpoint of side AB and E the midpoint of side AC. Which of the following pairs of ratios are equal? Give reasons for your answers.
   (a) $\frac{AB}{AC}, \frac{AD}{AE}$ (b) $\frac{AB}{AD}, \frac{AC}{AE}$ (c) $\frac{AD}{DE}, \frac{AB}{BC}$ (d) $\frac{AD}{AC}, \frac{AB}{AE}$
2. Would your answers in Problem 1 be different if you had started with a different triangle ABC? Why or why not?
3. Draw a right triangle ABC and let $\overline{DE}$ be a line segment parallel to $\overline{BC}$, where D is on $\overline{AB}$ and E is on $\overline{AC}$. Then answer the questions in Problem 1.
4. The angle of elevation of the top of a tree is 30° if the measurement is taken at a point 50 feet from the base of the tree. How tall is the tree? Draw a triangle to scale and use measurements to find this answer as was done in finding the height of a building in the first part of this section.
5. T and S are points at which trees are located on one side of a river and R is a point directly across the river from S, so that $\overleftrightarrow{RS}$ is perpendicular to $\overleftrightarrow{ST}$. With a sextant the measure in degrees of $\angle$ STR is found to be 60. The distance ST has been measured as 1000 feet. Draw a figure to scale, using 3 in. to represent 1000 feet. Find the distance from tree R to tree S.



5-2. Similar Triangles

Two triangles can be thought of as being similar if they are of the same "shape." A more careful definition is:

Definition: Two triangles are said to be similar if there is a one-to-one correspondence between the vertices so that corresponding angles are congruent and ratios of the measures of corresponding sides are equal.

Another way of saying this is: Suppose two triangles are labeled ABC and A'B'C'. Then they are similar if

1. $m(\angle A) = m(\angle A')$, $m(\angle B) = m(\angle B')$, $m(\angle C) = m(\angle C')$.
2. $\frac{AB}{A'B'} = \frac{AC}{A'C'} = \frac{BC}{B'C'}$.

Actually, if corresponding angles are congruent, ratios of the measures of corresponding sides are equal. Also if ratios of measures of corresponding sides are equal, corresponding angles are congruent. Hence if 1 holds, then 2 must hold; if 2 holds, then 1 must hold.

If two pairs of corresponding angles of two triangles are congruent, the third angles are also congruent. Thus there are two alternate definitions of similarity which seem to require less than the definition given above:

Alternate Definition 1: If ABC and A'B'C' are two triangles with the property that the angle at A is congruent to the angle at A', and the angle at B is congruent to the angle at B', then the triangles are similar.

Alternate Definition 2: If ABC and A'B'C' are two triangles with the property that

$$\frac{AB}{A'B'} = \frac{AC}{A'C'} = \frac{BC}{B'C'}$$

then the triangles are similar.

Exercises 5-2

1. In each of the following, ABC and A'B'C' are two similar triangles, in which A and A', B and B', C and C' are pairs of corresponding vertices. Fill in the blanks where it is possible. Where it is not possible, explain why.
   (a) $m(\angle A) = 30$, $m(\angle B) = 75$, $m(\angle A') = ?$, $m(\angle B') = ?$.
   (b) $AB = 3$, $AC = 4$, $A'B' = 6$, $A'C' = ?$.
   (c) $\frac{AB}{BC} = \frac{2}{3}$, $A'B' = 5$, $B'C' = ?$, $A'C' = ?$.
   (d) $\frac{BC}{AC} = \frac{4}{5}$, $A'C' = 3$, $A'B' = ?$, $B'C' = ?$.
   (e) $m(\angle A) = 30$, $m(\angle B) = 73$, $m(\angle A') = ?$, $m(\angle C') = ?$.
2. Find which of the following are true statements. Give reasons for your answers.

(a) If one acute angle of one right triangle is congruent to an acute angle of another right triangle, then the triangles are similar.

(b) If two sides of one triangle are congruent to two sides of another triangle, then the triangles are similar.

(c) If $\frac{AB}{A'B'} = \frac{AC}{A'C'}$ and $\frac{AB}{A'B'} = \frac{BC}{B'C'}$ then triangles ABC A'B'C' are similar.

(d) If $\frac{AB}{A'B'} = \frac{AC}{A'C'}$ then triangles ABC and A'B'C' are similar.

(e) If $\frac{AB}{A'B'} = \frac{AC}{A'C'}$ and $\frac{A'B'}{AB} = \frac{B'C'}{BC}$ then triangles ABC and A'B'C' are similar.

3. (a) If the corresponding angles of two quadrilaterals are congruent, must the ratios of the measures of corresponding sides be equal?

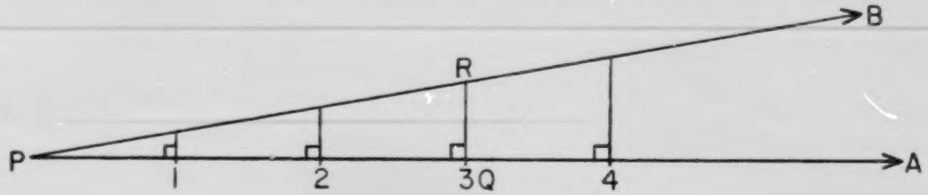
   (b) If the ratios of the measures of the corresponding sides of two quadrilaterals are equal, will corresponding angles be congruent?

5-3. The Trigonometric Ratios

One of the main problems in the branch of mathematics called trigonometry (Greek: "tri" means "three", "gon" means "angle", "trigon" means "triangle", and "metron" means "to measure"; thus "trigonometry" means "the measurement of triangles.") is to find the unknown measures of some parts of a triangle when the measures of the other parts are known.

One way that you might approach the problem would be as follows: You could draw a large number of triangles and measure the sides. Then you could tabulate your results. Of course, you would want to tabulate your results systematically so that it would be easy to locate information in the table. You might have on the first page a table showing the situation when $\angle QPR$ is of measure 1.

To work out the table, you would draw an angle of measure 1. Mark off lengths 1, 2, 3, 4 units, and so forth, on one side of the angle, draw perpendiculars, and measure the lengths of corresponding segments $\overline{QR}$. In the figure below, $\overline{PQ}$ is on ray $\overrightarrow{PA}$ and $\overline{PR}$ is on ray $\overrightarrow{PB}$. The figure is labeled for $PQ = 3$.



You might measure $\overline{QR}$ for all the cases from $PQ = 1$ to $PQ = 100$, or whatever limit you may choose.

Then on Page 2 you could make a similar table showing what happens when angle QPR is 2° in measurement. On the next page you would have a table for the case where $\angle QPR$ is 3° in measurement. If you continue in this way, with $\angle QPR$ increasing 1° in measurement at a time, how many pages will you need in the book? If there are 100 entries on each page, how many entries will there be in all? If it takes you 1 minute per entry to make the construction and measure the lengths, how many hours would you need to make the table? Fortunately these measurements have been made for us and are recorded in a table.

The ratio,

$$\frac{PQ}{PR} = \frac{\text{measure of the length of adjacent side}}{\text{measure of the length of hypotenuse}},$$

is called the cosine of $\angle QPR$. It is abbreviated like this:

$$\cos \angle QPR.$$

Other useful ratios formed from the lengths of the sides of the triangle PQR are the sine (abbreviated sin) and the tangent (abbreviated tan).

$$\sin \angle QPR = \frac{QR}{PR} = \frac{\text{measure of the length of opposite side}}{\text{measure of the length of hypotenuse}}$$

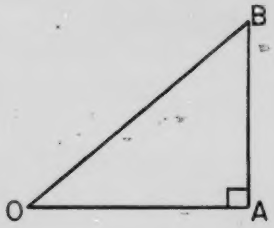
and $$\tan \angle QPR = \frac{QR}{PQ} = \frac{\text{measure of the length of opposite side}}{\text{measure of the length of adjacent side}}$$

A part of a table of trigonometric ratios would look like this.

m(∠ QPR	sin ∠ QPR	cos ∠ QPR	tan ∠ QPR
20	0.34	0.94	0.36
40	0.64	0.77	0.84
60	0.87	0.50	0.58
70	0.94	0.34	2.75

You can use the table to solve problems like the following:

Sample Problem: In the right triangle AOB,



The measure of ∠ AOB is 40, and the length of OA is 4 inches. Find AB and OB.

Solution: First let us find AB. The side $\overline{AB}$ is opposite the given angle AOB, while the known side $\overline{OA}$ is the adjacent side. The only ratio involving the opposite side and the adjacent side is the tangent.

$$\tan \angle AOB = \tan 40^\circ = \frac{AB}{OA} = \frac{AB}{4} .$$

If the measure in degrees of ∠ AOB is 40, you can use the notation

$\tan \angle AOB = \tan 40^\circ$, since it is said that ∠ AOB is a 40° angle. Look up $\tan 40^\circ$ in the table and find

$$\tan 40^\circ \approx 0.84.$$

$\approx$ means "approximately equal to". You will obtain, then, the relation

$$0.84 \approx \frac{AB}{4} .$$

This relation can be solved for the unknown AB.

$$AB \approx 4(0.84) \approx 3.4$$

The measurement of $\overline{AB}$ is 3.4 inches, approximately.

By similar reasoning, OB can be found by the following steps:

$$\cos 40^\circ = \frac{OA}{OB} = \frac{4}{OB},$$

$$\cos 40^\circ \approx 0.77,$$

$$\frac{4}{OB} \approx 0.77,$$

$$4 \approx 0.77\,(OB) \quad \text{or} \quad 0.77\,(OB) \approx 4$$

$$OB \approx \frac{4}{0.77} \approx 5.2$$

The length of $\overline{OB}$ is approximately 5.2 inches. Check your answers by use of the Pythagorean property.

$$(OA)^2 + (AB)^2 = (OB)^2;$$

$$(OA)^2 + (AB)^2 \approx 4^2 + (3.4)^2 \approx 27.6$$

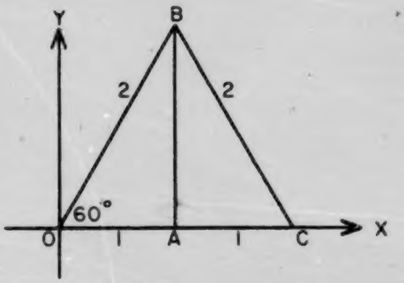
$$(OB)^2 \approx (5.2)^2 \approx 27.0;$$

$$\text{Is } 27.6 \approx 27.0 \text{ ?}$$

Your results will not check exactly because approximations to the values of cos 40° and tan 40° were used.

It so happens that for certain angles such as 30°, 45°, and 60°, trigonometric ratios can be found by reasoning instead of measurement.

Consider an equilateral triangle OBC whose sides are 2 units long. The angles of this equilateral triangle are 60° in measurement. Join the vertex B to the midpoint A of the segment $\overline{OC}$. Since the corresponding sides of the triangles OAB and CAB are equal in length, these triangles are congruent. Therefore the corresponding angles ∠ ABO and ∠ ABC are equal in measurement. Since ∠ OBC is 60° in measurement, then ∠ ABO and ∠ ABC are 30° in measurement. In



the same way, you can see that angles $\angle OAB$ and $\angle CAB$ are equal in measurement, and so must be right angles. Why?

Therefore triangle OAB is a right triangle.

$$\cos 60^\circ = \cos \angle AOB = \frac{OA}{OB} = \frac{1}{2}\ .$$

In order to find the value of $\sin\ 60^\circ$ and $\tan\ 60^\circ$, find AB, which shall be called y. By the Pythagorean property,

$$(OA)^2 + (AB)^2 = (OB)^2.$$

Thus

$$1 + y^2 = 4,$$

$$y^2 = 3,$$

$$y = \sqrt{3} \approx 1.7321.$$

(Check this approximation for $\sqrt{3}$ by calculating $(1.7321)^2$.)

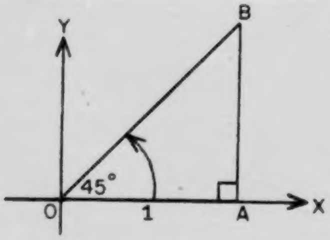
$$\sin 60^\circ = \frac{AB}{OB} = \frac{\sqrt{3}}{2} \approx 0.8660$$

and

$$\tan 60^\circ = \frac{AB}{OA} = \frac{\sqrt{3}}{1} \approx 1.7321.$$

Exercises 5-3

1. Find the values of $\sin\ 30^\circ$, $\cos\ 30^\circ$, and $\tan\ 30^\circ$.
2. Using the figure



Find the trigonometric ratios for an angle of measurement 45°. You may calculate the ratios by using $\sqrt{2} \approx 1.4142$.

3. In Problem 2, you will find $\sin\ 45^\circ = \frac{1}{\sqrt{2}}$. You could then determine a decimal expression for $\frac{1}{\sqrt{2}}$ by dividing 1 by 1.4142. If the computation seems tedious you might think about another numeral for $\frac{1}{\sqrt{2}}$

$$\frac{1}{\sqrt{2}} \cdot \frac{\sqrt{2}}{\sqrt{2}} = \frac{\sqrt{2}}{2} \text{ or } \frac{1}{2}\sqrt{2}.$$

It is easier to divide 1.4142 by 2 than to divide 1 by 1.4142. Find sin 45° and cos 45°, by computing $\frac{1}{2}\sqrt{2}$.

4. Find the ratios: $\frac{\sin 60^\circ}{\sin 30^\circ}$ and $\frac{\tan 60^\circ}{\tan 30^\circ}$

State in your own words why you think these ratios are not 2.

5. A regular hexagon is inscribed in a circle of radius 10 inches.

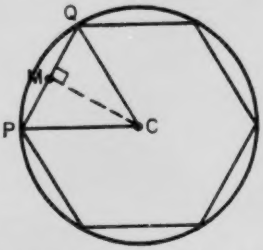
(a) What is the measure of $\angle$ PCQ?

(b) What is the measure of $\angle$ CPQ?

In (c) and (d) use an approximation for $\sqrt{3}$ and express your answers as decimals correct to 0.01.

(c) Find CM

(d) Find PQ



In problems 6 - 9 use the table in the earlier part of this section.

6. In the table, compare the values of sin 20° and cos 70°. Is there any reason for what you notice?

7. Examine the sine column in the table. As $\angle$ AOB increases in measurement from 20° to 70°, does sin $\angle$ AOB increase or decrease? What about cos $\angle$ AOB? What about tan $\angle$ AOB? Look at the table to answer the question about tan $\angle$ AOB.

8. Using the table, find

$$(\sin 20^\circ)^2 + (\cos 20^\circ)^2.$$

What answer should you get? Why? Check by finding

$$(\sin 30^\circ)^2 + (\cos 30^\circ)^2$$

and

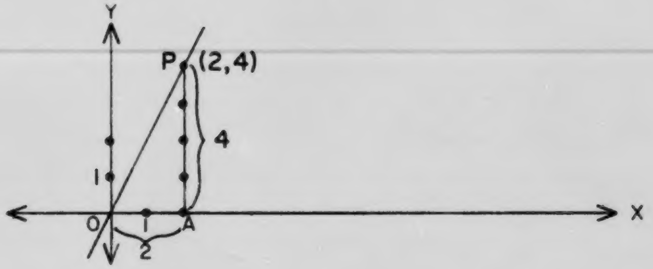
$$(\sin 60^\circ)^2 + (\cos 60^\circ)^2.$$

9. Use the values in the table and calculate $\frac{\sin 20^\circ}{\cos 20^\circ}$

Compare your result with tan 20°. What do you notice? What is the reason?

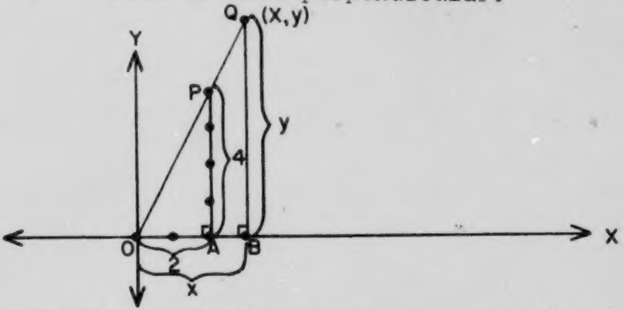
5-4. Slope of a Line

On a sheet of graph paper draw the line joining the origin to the point P whose coordinates are (2,4).



Where does this line intersect the vertical lines through (3,0), (4,0), and (2,0)? For each of these intersection points (x,y) find $\frac{y}{x}$. What do you notice?

Take _any_ point Q other than O on $\overleftrightarrow{OP}$. Let the coordinates of Q be (x,y). Drop the perpendicular from Q to the X-axis, and let B be the foot of this perpendicular.



What do you conclude about the ratios

$$\frac{BQ}{OB} \text{ and } \frac{AP}{OA} \text{ ?}$$

Since $\frac{AP}{OA}$ is $\frac{4}{2}$, or 2, and $\frac{BQ}{OB} = \frac{y}{x}$, then $\frac{y}{x}$ for any point (other than O) on line $\overleftrightarrow{OP}$ is 2.

If $\frac{y}{x} = 2$, then $y = 2x$.

Plot the points $(^-1,^-2)$, $(^-2,^-4)$, and $(^-3,^-6)$ on your graph. Do they lie on the line OP? For each of these points calculate the ratio $\frac{y}{x}$. For any point on $\overleftrightarrow{OP}$, $y = 2x$.

On another sheet of graph paper draw the line which passes through the origin (0,0) and the point P' with coordinates (2,3). Do the points (4,6), (6,9), and $(1,\frac{3}{2})$ lie on $\overleftrightarrow{OP'}$? For each of these points, what is the ratio $\frac{y}{x}$? What property of similar triangles tells us that

$$\frac{y}{x} = \frac{3}{2} \text{ (provided that } x \neq 0) \text{ ?}$$

This could be written

$$y = \frac{3}{2}x \text{ .}$$

In general, if P is a point for which the ratio $\frac{y}{x}$ is m, then the following statement is true:

If Q, with coordinates (x,y), is any point on $\overleftrightarrow{OP}$ then $y = mx$.

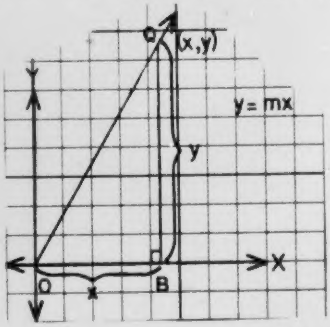
Thus, the equation of any line through the origin (except the y-axis!) has the form

$$y = mx,$$

where m is a certain number.

The number m is called the slope of the line. The slope of the line, $y = \frac{3}{2}x$, is $\frac{3}{2}$. The slope is a measure of the steepness of the line. Look again at any line $y = mx$, $m \neq 0$.

Consider any point Q(x,y) on the line. Drop the perpendicular from Q to the X-axis and let B be the foot of this perpendicular. Since $y = mx$,



$$\frac{y}{x} = m = \text{slope of line.}$$

Also in the right triangle OBQ,

$$\tan \angle BOQ = \frac{\text{measure of opposite side}}{\text{measure of adjacent side}} = \frac{y}{x} \text{ .}$$

We conclude, then, that

$$\text{slope} = m = \frac{y}{x} = \tan \angle BOQ.$$

We can think of the slope geometrically as the tangent of the angle BOQ which the line makes with the X-axis.

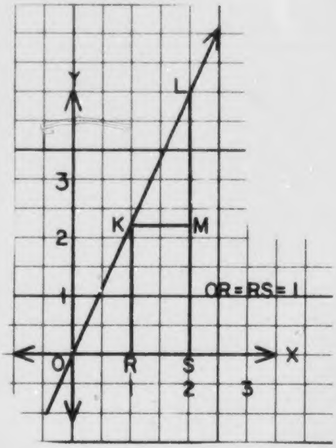
Exercises 5-4

1. Find the equations of the lines joining the origin to each of the following points: (4,1), (3,1), (1,1), (1,2), (1,5).

2. Consider the line $\overleftrightarrow{OP'}$ discussed in the text, where P' is (2,3). What is the tan $\angle$ AOP', where $\angle$ AOP' is the angle the line makes with the x-axis?

3. (a) Draw the graph of $y = \frac{7}{3}x$.

   (b) Mark the point R on the X-axis to the right of O such that OR = 1. Construct the perpendicular to the X-axis at R. Let K be the point of intersection of this perpendicular and the graph in (a).



   (c) What is RK in terms of OR? Write a numeral for RK. Write the coordinates of K.

   (d) Draw OS = 2 and then draw $\overline{LS}$ perpendicular to the x-axis, where L is a point on the graph in (a).

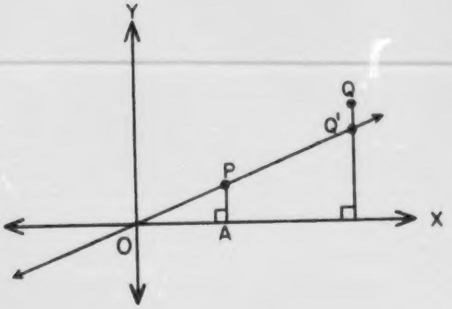
   (e) What is the measure of SL?

   (f) What is the ratio $\frac{RK}{OR}$? the ratio $\frac{SL}{OS}$?

4. Let P be a point with coordinates (a,b), where $\frac{b}{a} = m$. The coordinates of all points on $\overleftrightarrow{OP}$ satisfy the equation.

$$y = mx.$$

If Q is any point (c,d) whose coordinates satisfy this equation, then, $d=mc$ is a true sentence. Draw the line through Q perpendicular to the X-axis, and let Q', with coordinates (c,d') be the point where this perpendicular intersects the line $\overleftrightarrow{OP}$.



(a) Why must $d' = mc$ be a true sentence?

(b) What is true of d and d'? What is true of Q and Q'?

(c) What have you proved in (a) and (b)?

5. Graph the lines $y = \frac{2}{3}x$, $y = \frac{1}{4}x$, and $y = 4x$ on the same set of axes. Use four values of x for each graph. Find the slopes of these lines.

6. Find the slopes of the lines joining the following pairs of points:

(a) $(0,0)$ and $(1,3)$

(b) $(0,0)$ and $(2,3)$

(c) $(0,0)$ and $(\frac{1}{2}, 5)$

*(d) $(\frac{1}{2}, 2)$ and $(2,8)$

7. Draw a rectangle with one vertex at $(0,0)$ and its interior completely in the first quadrant. The measures of the sides of the rectangle are to be 3 and 5. Find the slope of its diagonal from $(0,0)$. (There are two possible answers.)

8. A road rises 10 feet over a distance of one mile. What is the slope of the road? Consider the one mile to be a horizontal distance.

9. Choose an appropriate scale on coordinate axes so that you can compare the graphs of $y = 7x$ and $y = 8x$. How does the measure of the angle determined by these lines compare with the measure of the angle formed by lines of slopes 1 and 2?

5-5. Reading Trigonometric Ratios From a Table

To read a complete table of trigonometric ratios you use the ratio-headings at the top for the angles whose measurement is found on the left, and the headings at the bottom for the angles whose measurement is found on the right. For example, to find sin 20° in the table, first look for 20°. It is in the column on the left. Look for the column with "Sine" at the top. The number in the 20° - row and the Sin-column is 0.3420. Sin $20^\circ \approx 0.3420$. If you want to know cos 70°, you will find 70° in the column on the right. Therefore you will use the column headings at the bottom. Look for the number in the 70° — row and the column with "Cosine" at the bottom, and find cos $70^\circ \approx 0.3420$. This is the same number as before. This is as it should be, for angles of 20° and 70° are complementary, so that sin 20° = cos 70°. Why?

Just as the sine of the complement of the angle is called the cosine of the angle, so the tangent of the complement of an angle is called the cotangent of the angle. For instance,

$$\cot 70^\circ = \tan 20^\circ.$$

The symbol "cot" is used as an abbreviation for "cotangent." You notice that the "co" in cosine and the "co" in cotangent are suggested by the "co" in complementary.

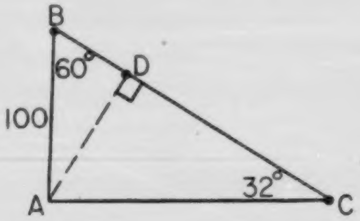
The table readings are given as four-place decimals. Most of these readings have been rounded to the nearest 0.0001. They are only approximate values and hence we use sin $20^\circ \approx 0.3420$. We say "most" because some do not need to be rounded, like cos $60^\circ = 0.5$.

TRIGONOMETRIC RATIOS

Angle	Sine	Tangent	Cotangent	Cosine	
1°	.0175	.0175	57.290	.9998	89°
2°	.0349	.0349	28.636	.9994	88°
3°	.0523	.0524	19.081	.9986	87°
4°	.0698	.0699	14.301	.9976	86°
5°	.0872	.0875	11.430	.9962	85°
6°	.1045	.1051	9.5144	.9945	84°
7°	.1219	.1228	8.1443	.9925	83°
8°	.1392	.1405	7.1154	.9903	82°
9°	.1564	.1584	6.3138	.9877	81°
10°	.1736	.1763	5.6713	.9848	80°
11°	.1908	.1944	5.1446	.9816	79°
12°	.2079	.2126	5.7046	.9781	78°
13°	.2250	.2309	4.3315	.9744	77°
14°	.2419	.2493	4.0108	.9703	76°
15°	.2588	.2679	3.7321	.9659	75°
16°	.2756	.2867	3.4874	.9613	74°
17°	.2924	.3057	3.2709	.9563	73°
18°	.3090	.3249	3.0777	.9511	72°
19°	.3256	.3443	2.9042	.9455	71°
20°	.3420	.3640	2.7475	.9397	70°
21°	.3584	.3839	2.6051	.9336	69°
22°	.3746	.4040	2.4751	.9272	68°
23°	.3907	.4245	2.3559	.9205	67°
24°	.4067	.4452	2.2460	.9135	66°
25°	.4226	.4663	2.1445	.9063	65°
26°	.4384	.4877	2.0503	.8988	64°
27°	.4540	.5095	1.9626	.8910	63°
28°	.4695	.5317	1.8807	.8829	62°
29°	.4848	.5543	1.8040	.8746	61°
30°	.5000	.5774	1.7321	.8660	60°
31°	.5150	.6009	1.6643	.8572	59°
32°	.5299	.6249	1.6003	.8480	58°
33°	.5446	.6494	1.5399	.8387	57°
34°	.5592	.6745	1.4826	.8290	56°
35°	.5736	.7002	1.4281	.8192	55°
36°	.5878	.7265	1.3764	.8090	54°
37°	.6018	.7536	1.3270	.7986	53°
38°	.6157	.7813	1.2799	.7880	52°
39°	.6293	.8098	1.2349	.7771	51°
40°	.6428	.8391	1.1918	.7660	50°
41°	.6561	.8693	1.1504	.7547	49°
42°	.6691	.9004	1.1106	.7431	48°
43°	.6820	.9325	1.0724	.7314	47°
44°	.6947	.9657	1.0355	.7193	46°
45°	.7071	1.0000	1.0000	.7071	45°
	Cosine	Cotangent	Tangent	Sine	Angle

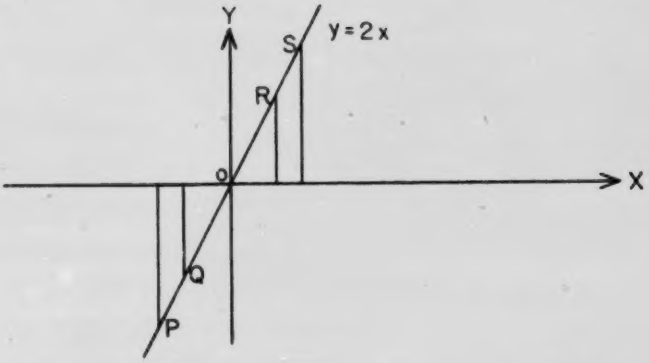
Exercises 5-5

1. Use the table to find the following:
   - (a) $\sin 10^\circ$
   - (b) $\tan 10^\circ$
   - (c) $\sin 41^\circ$
   - (d) $\cos 63^\circ$
   - (e) $\sin 82^\circ$
   - (f) $\tan 40^\circ$
   - (g) $\cos 17^\circ$
   - (h) $\tan 60^\circ$
   - (i) $\tan 70^\circ$
   - (j) $\cot 88^\circ$
2. Check the statements below by studying the numbers in the Table. Do you agree with the statements?
   - (a) The sine of the angle in the Table is always between 0 and 1.
   - (b) The sine of an angle increases with the size of the angle from 1° to 89°.
   - (c) The sine of an angle less than 30° is less than $\frac{1}{2}$.
   - (d) The differences between consecutive Table readings varies throughout the Table.
   - (e) The difference between the sines of two consecutive angles is greater for smaller consecutive angles than for larger consecutive angles.
3. State properties for the tangent which are similar to those given in Problem 2 for the sine.
4. Find the following products:
   - (a) $100 \sin 32^\circ$
   - (b) $81 \tan 48^\circ$
   - (c) $0.27 \sin 73^\circ$
   - (d) $0.05 \tan 80^\circ$.
5. In the figure at the right $\angle ABC$ has measure 60 and $\angle ACB$ has measure 32, in degrees, and $AB = 100$. Find (a) AD and (b) BC.



5-6. Kinds of Variation

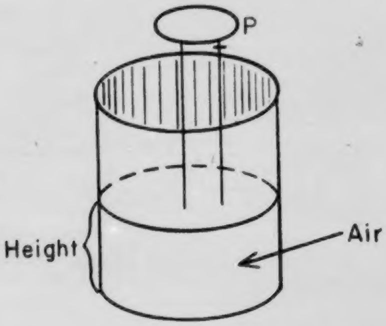
Let us think again of the equation $y = 2x$ and its graph.



If perpendiculars are dropped from the points P,Q,R,S on the graph to the X-axis, four similar right triangles are formed. The ratio of the y-coordinate to the x-coordinate of each of these points is 2, and 2 is called the slope of the line. There is a relation between the y- and x-coordinates of points on this graph and this relation is expressed by the equation $y = 2x$.

In the world around you there are many situations where two quantities are related. When you buy peanuts, the amount you pay depends on how much you buy. If you hang a mass on a spring balance, the distance that the spring stretches is related to the weight of the mass.

Here there is a picture of a cylinder filled with air. Pressure can be exerted by placing a weight on the platform P which is



connected to a piston in the cylinder. The volume occupied by the air depends on the pressure, which, in turn, depends on the weight placed at P.

The distance through which a falling body has traveled since it was dropped depends on the time elapsed since it was dropped. The force with which two planets attract each other, according to Newton's law of gravitation, depends on the distance between them.

In all of these cases when one of the quantities changes, then the other one also changes in a certain definite way. In the rest of this chapter you will study some of the simplest properties related to these changes, which are sometimes called laws of variation.

Exercises 5-6

1. Suppose peanuts cost \$.60 per pound. Make a table showing the cost of various amounts of peanuts:

Amount in pounds	0	1	2	3	4	5	6	7	8	9	10
Cost in dollars											

Make a graph showing how the cost is related to the weight. Let w be the weight in pounds and c be the cost in dollars. What simple geometrical figure is formed by the points in the graph?

2. A girl measured the distances through which a spring stretched when she hung masses of various weights on it. Here is a table of her observations:

Weight in pounds	0	1	2	3	4
Stretch in inches	0	$\frac{3}{4}$	$1\frac{1}{2}$	$2\frac{1}{4}$	3

Make a graph showing the relation between the weight w in pounds and the stretch s in inches.

Find a formula for s in terms of w which fits her observations. Using your formula, or your graph, predict the stretch in the spring if she hangs a mass of weight $2\frac{1}{2}$ pounds on it.

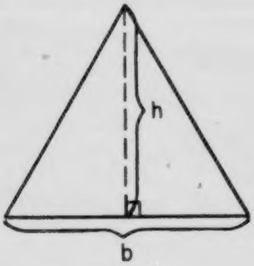
3. A boy placed masses of various weight on the platform P of the piston. He used a pressure gauge to measure the pressure p in pounds per square inch of the air in the cylinder. He also measured the height of the piston each time, and cal-

culated the measure v of the volume, in cubic inches, of air in the cylinder. Here are his results:

p	15	20	25	30
v	250	187.5	150	125

Make a graph showing the relation between p and v. Predict the measure of the volume if $p = 40$.

4. Make a table showing the relation between the measure h of the length of the altitude and the measure b of the length of the base in an equilateral triangle. Calculate your results correct to 1 decimal place.



b	2	4	6	8	10
h					

This table can be completed by measurement. Construct equilateral triangles of base 2 units, 4 units, etc. Your results will be more accurate if your triangles are not too small. Make a graph showing the relation between b and h. What simple geometric figure is formed by the graph?

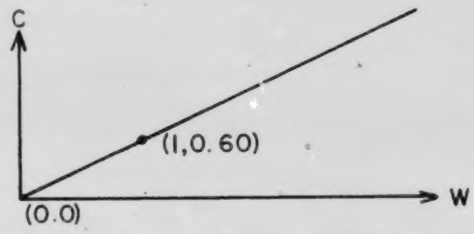
5. Give a formula expressing h in terms of b in Problem 4. You may wish to use $\tan 60^\circ = h \div \frac{b}{2}$.

6. Make a table showing the relation between the measure b of the length of the base and the measure A of the area of an equilateral triangle. Use the values of b which you used in your table for Problem 4. Make a graph showing the relation between b and A. Give a formula for A in terms of b.

5-7. Direct Variation

In Problem 1 of the last set of exercises the relation between cost (c) and weight (w) would be $c = (0.60)w$.

When you graphed this relation, you obtained a straight line through the origin:



It is said that c varies directly as w, or that the cost is proportional to the weight and that 0.60 is the constant of proportionality. In this relation, the ratio of the measure of the cost to the measure of the weight is always 0.60. Because it does not change, it is called a constant.

If you drive along a straight road at the speed of 50 miles per hour, how far do you go in 1 hour? 2 hours? 3.5 hours? t hours? The measure d of the distance traveled is given in terms of the measure t of the time by the formula:

$$d = 50t.$$

The measure of the distance is a constant times the measure of the time. The ratio $\frac{d}{t}$ of the measure of the distance to the measure of the time is a constant. The distance varies directly as the time. The distance is proportional to the time. The constant of proportionality is 50.

In the graph of the equation $y = 2x$, the y-coordinates of the points on the graph are a constant times the x-coordinates. In other words, y varies directly as x. The y-coordinate is proportional to the x-coordinate and the constant of proportionality is 2, the slope of the line.

According, to Hooke's law of elasticity, the amount that a spring (see Problem 2 of Section 5-6) stretches is proportional to the weight of the object hung on it. Suppose you know that when a mass having a weight of 2 pounds is hung on it the

the spring stretches 3 inches. How much stretch would be produced by an object weighing 5 pounds?

Solution: Hooke's law may be expressed by means of the equation:

$$s = k \cdot w,$$

where k is some constant of proportionality, and s and w are the measures of the stretch in inches and the weight in pounds, respectively. According to the given information, when $w = 2$, $s = 3$. Applying the above equation, you will find $3 = (k)(2)$. All you have to do now is to solve this equation for the unknown constant k, and then calculate s from the formula $s = kw$, when $w = 5$. Finish the problem.

Exercises 5-7

1. Write a sentence in mathematical terms about the total cost, t cents, of n gallons of gasoline at 32 cents per gallon. In this statement the cost may also be stated as d dollars. Write the sentence a second way, using d dollars.
2. Suppose the gasoline you bought cost 33.9 cents a gallon. Write a sentence showing the cost, c cents, of g gallons of this gasoline.
3. If your pace is normally about 2 feet, how far will you walk in n steps? Use d feet for the total distance and write the formula. If n increases, can d decrease at the same time?
4. Write a formula for the number i of inches in f feet. As f decreases what happens to i?
5. State the value of the constant, k, in each of the equations you wrote for Problems 1 through 4.
6. Can you write the equation in Problem 3 in the form $\frac{d}{n} = 2$? What restriction does this form place on n?
7. Find k if y varies directly as x, and y is 6 when x is 2.
8. Find k if y varies directly as x, and y is 3 when x is 12.

9. Sometimes it is required to write an equation from a set of number pairs. From the information in the table, does it appear that a varies directly as b? Why? What equation appears to relate a and b?

a	b
20	2
50	5
$^{-}70$	$^{-}7$
100	10
$^{-}110$	$^{-}11$

10. Suppose that d varies directly as t and that when t is 6, d is 240. Write the equation relating d and t.

11. Use the relation $y = \frac{3x}{2}$ to supply the missing values in the following ordered pairs: $(^{-}4, \ )$; $(^{-}3, \ )$; $(^{-}2, \ )$; $(^{-}1, \ )$; $(0, \ )$; $(2, \ )$; $(5, \ )$. Plot the points on graph paper.

12. In the relation of Problem 11, when the number x is doubled, is the number y doubled? When x is halved, what happens to y? When the number y is multiplied by 10 what happens to the number x? Are your statements true for negative values of x and y?

13. In the equation $y = kx$ what happens to the number x if y is halved? What happens to y if x is tripled?

5-8. Inverse Variation

Suppose you have 10 gallons of punch for a party, and you want to be perfectly fair to your guests and serve each one exactly the same amount. How does the amount for each guest vary with the number n of guests? If the number of guests is doubled, how is the amount that each one gets changed? Let p be the number of gallons of punch per guest. Then the total amount of punch, which is 10 gallons, is equal to:

(the number of gallons per guest) × (number of guests).

The relation between p and n can be expressed by means of the equation

$$pn = 10.$$

Give a formula for p in terms of n.

In this case p varies inversely as n or p is inversely proportional to n. In general, it is said that one quantity varies inversely as another if their product is a non-zero constant. If the number substituted for n is multiplied by 3, that is, it is tripled, what is true of the corresponding values of p which make $pn = 10$ a true statement?

Boyle's law states that, at constant temperature, the volume of a gas is inversely proportional to the pressure or, in mathematical language,

$$p \cdot v = k.$$

Find the constant in the experiment and check to see whether this equation fits the observations. (See Problem 3 of Exercises 5-6).

Exercises 5-8

1. (a) The table below, as it is now filled in, shows two possible ways in which a distance of 100 miles can be traveled. Complete the table.

Rate (mi. per hr.)	10	20	25	50	60	75	80	100
Time (hours)	10	5						

   (b) From part (a), use r for the number of miles per hour and t for the number of hours and write an equation connecting r and t and 100.

   (c) When the rate is doubled what is the effect upon the corresponding time value?

   (d) When t increases in $rt = 100$ what happens to r?

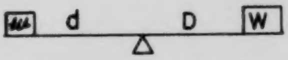
2. (a) Suppose you have 240 square patio stones (flagstones). You can arrange them in rows to form a variety of rectangular floors for a patio. If s represents the number of stones in a row and n represents the number of rows, what are the possibilities? Fill in a table like this one.

Total Number of Stones: 240

Number of stones in a row	10	12	15	16	30	40
Number of rows	24					

   (b) Write an equation connecting n, s, and 240. (If you cannot cut any of the stones, what can you say about the kind of numbers n and s must be?)

3. (a) A seesaw will balance if $wd = WD$ when an object weighing w pounds is d feet from the fulcrum and on the other side an object weighing W pounds is D feet from the fulcrum. If $WD = 36$, find D when W is 2, 9, or 18 and find W when D is 1, 6, 12.



   (b) What can you say about corresponding values of W as D is doubled if wd remains constant? As numbers substituted for W increase, what can you say about corresponding values of D provided wd remains constant?

4. Write an equation connecting rate of interest r and the number of dollars on deposit p with a fixed interest payment of $200 per year. Discuss how corresponding values of r are affected as different numbers are substituted for p. If the interest rate were doubled how much money would have to be on deposit to give $200 interest per year?

5. Give the constant of proportionality in each of the Problems 1 through 4.

6. State your impression of the difference between direct variation and inverse variation.

7. Find k if y varies inversely as x and if y is 6 when x is 2.

8. Find k if x varies inversely as y and if y is 10 when x is $\frac{1}{2}$.

9. From the information in the table does it appear that a varies inversely as b? Explain your answer.

a	$^-4$	$^-1$	1	3	8	19	41
b	$^-8$	$^-2$	2	6	16	38	82

10. Study the number pairs which follow: $(^-2, 8)$; $(^-1, 2)$; $(0,0)$; $(1,2)$; $(2,8)$; $(3,18)$; $(4,32)$.
    (a) Does it appear that y varies directly as x?
    (b) Does it appear that y varies inversely as x?

11. (a) Supply the missing values in the table below where $xy = 18$.

x	$^-4$	$^-3$	$^-2$	$^-1$	1	2	3	4	5	6	7	8	9	18
y														

(b) Is it possible for x or y to be zero in $xy = 18$? Why?

(c) Plot on graph paper the points whose coordinates you found in (a) and draw the curve. You may wish to find more number pairs to enable you to draw the curve more easily.

Notice that in the discussion of direct and inverse variations, the letters x and y may be used interchangeably. In $y = kx$, if k is not zero, it is said that x varies directly as y and y varies directly as x. In $xy = k$, k cannot be zero and it is said that x varies inversely as y or that y varies inversely as x. In these statements x and y can represent different pairs of numbers while k represents a constant, that is, a fixed number. In the general equation the letter "k" is used rather than a particular numeral, in order to include all possible cases. For $x \neq 0$, the equation $xy = k$ may be written $y = k \cdot \frac{1}{x}$ which says that y varies directly as the reciprocal of x.

Occasionally you may see direct variation represented by the statement $\frac{y}{x} = k$. There are times when this form is useful but from your work with zero you know that $\frac{y}{x} = k$ excludes the possibility of x being zero.

The graphs of $y = kx$ and $xy = k$ include points with negative coordinates. In many problems it doesn't make sense for x or y to be negative. In the problem of serving punch at your party, the number n of guests must be a counting number. In such cases the equation is not a completely correct translation of the relation into mathematical language. The correct translation of your punch-at-the-party problem is the number sentence

(1) "$pn = 10$ and n is a counting number."

The correct translation of Boyle's law is

(2) "$pv = k$ and $p > 0$ and $v > 0$."

When you graph the number sentence (1) you obtain a set of isolated points in the first quadrant. The graph of the relation (2) is the branch of the hyperbola

$$pv = k$$

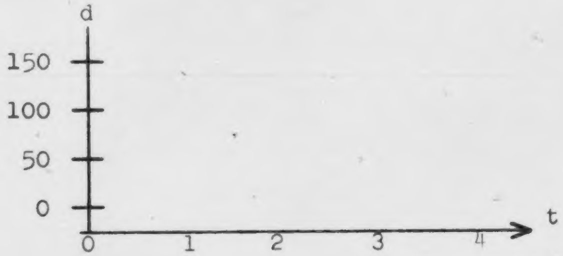
which lies in the first quadrant.

5-9. Other Types of Variation

If you make a table of the measure d of the distance in feet through which an object falls from rest in t seconds, you will obtain results like this:

t	0	1	2	3	4
d	0	16	64	144	256

Sketch a graph of the relation between d and t. Since the numbers for d are large you may wish to use a different unit on the d-axis from that used on the t-axis.



You obtain a part of a curve called a parabola.

If the number t is doubled, by how much is the number d multiplied? If t is tripled, by how much is d multiplied? As you see, d varies directly as t^2. The relation can be expressed by means of the equation

$$d = kt^2,$$

where k is constant. What is this constant? How far does the body fall in 10 seconds?

Exercises 5-9

1. (a) What is the area of each face of a cube whose sides have length 2 inches? How many faces are there? What is the total surface area, the total area of all faces?

   (b) Make a table showing the relation between the lengths of each side of a cube and the surface area.

s	1	2	3	4	5	6
A						

2. Let S be the measure of the area in square centimeters of a square with edges e centimeters long.
   - (a) Find an equation connecting S and e.
   - (b) Tell how S varies with e.
   - (c) Plot the graph of the equation you found in (a). Use values of e from 0 to 15 and choose a convenient scale for the values of S.
   - (d) From the graph you drew in (c), find:
     - (1) The area of a square with edges 3 cm. long.
     - (2) The length of the edges of a square of area 64 square centimeters.
     - (3) The area of a square with edges 5.5 cm. long.
     - (4) The length of the edges of a square of area 40 sq. cm.
   - (e) From the equation you found in (a), find:
     - (1) The area of a square with edges 3 cm. long.
     - (2) The area of a square with edges 5.5 cm. long.
3. If E is proportional to the square of v and E is 64 when v is 4, find:
   - (a) an equation connecting E and v.
   - (b) the value of E when $v = 6$.
   - (c) the value of v when $E = 16$.
4. Suppose grass seed costs 70 cents per pound, and one pound will sow an area of 280 sq. ft.
   - (a) How many pounds of seed will be needed to sow a square plot 10 feet on a side?
   - (b) How much will it cost to buy seed to sow a square plot 10 feet on a side?
   - (c) If C cents is the cost of the seed to sow a square plot s feet on a side, find an equation connecting C and s.
   - (d) How much will it cost for seed to sow a square plot 65 feet on a side?
   - (e) If $15.00 is available for seed, can enough be bought to sow a square plot 75 feet on a side?
5. A ball is dropped from the top of a tower. The distance, d feet, which it has fallen varies as the square of the time, t seconds, that has passed since it was dropped.

(a) From the information above, what equation can you write connecting d and t ?

(b) Find how far the ball falls in the first 3 seconds.

(c) If you are also told that the ball falls 144 feet in the first 3 seconds, write an equation connecting d and t.

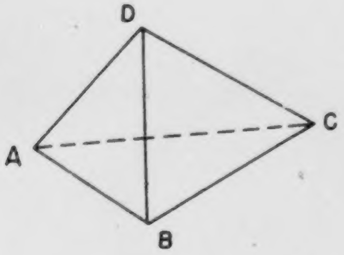
(d) Using the equation you wrote in (c), find how far the ball falls in the first 5 seconds.

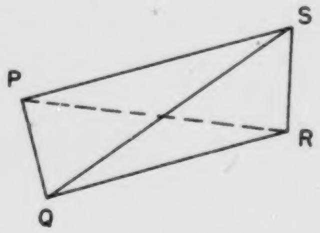
Chapter 6

NON-METRIC GEOMETRY II

6-1. Tetrahedrons

A geometric figure of a certain type is called a tetrahedron. A tetrahedron has four vertices which are points in space. The drawings below represent tetrahedrons. (Another form of the word "tetrahedrons" is "tetrahedra." ("Tetra" is the Greed word for four.)



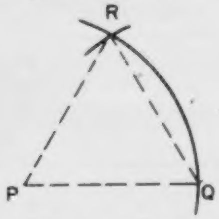


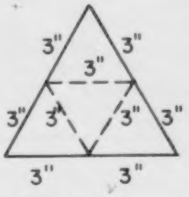
The points A, B, C, and D are the vertices of the tetrahedron on the left. The points P, Q, R, and S are the vertices of the one on the right. The four vertices of a tetrahedron are not in the same plane. The word "tetrahedron" refers either to the surface of the figure or to the "solid" figure, i.e. the figure including the interior in space. From some points of view the distinction is not important, from others it is. One can name a tetrahedron by naming its vertices. Parentheses shall usually be put around the letters like (ABCD) or (PQRS) in naming tetrahedrons. The vertices may be named in any order. Later this notation shall be used to mean specifically "solid tetrahedron", the union of the surface and its interior.

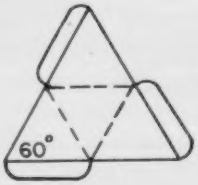
The segments $\overline{AB}$, $\overline{BC}$, $\overline{AC}$, $\overline{AD}$, $\overline{BD}$, and $\overline{CD}$ are called edges of the tetrahedron (APCD). Sometimes the notation (AB) or (BA) will be used to mean the edge $\overline{AB}$. What are the edges of the tetrahedron (PQRS)?

Any three vertices of a tetrahedron are the vertices of a triangle and lie in a plane. A triangle has an interior in the plane in which its vertices lie (and in which it lies). Let us use (ABC) to mean the triangle ΔABC together with its interior. In other words, (ABC) is the union of ABC and its interior. The sets (ABC), (ABD), (ACD), and (BCD) are called the faces of the tetrahedron (ABCD). What are the faces of the tetrahedron (PQRS)?

You will be asked to make some models of tetrahedrons in the exercises. The easiest type of tetrahedron of which to make a model is the so-called regular tetrahedron. Its edges are all the same length. (Length or measurement is introduced here only for convenience in making some uniform models. This chapter deals fundamentally with non-metric or "no-measurement" geometry.) On a piece of cardboard or stiff paper construct an equilateral triangle of side 6". You can do this with a ruler and compass or with a ruler and protractor.

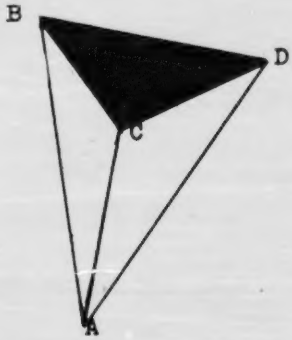






Now mark the three points that are halfway between the pairs of vertices. Cut out the large triangular region. Carefully make three folds or creases along the segments joining the "halfway" points. You may use a ruler or other straightedge to help you make these folds. Your original triangular region now looks like four smaller triangular regions. Bring the original three vertices together above the center of the middle triangle. Fasten the loose edges together with tape or paper and paste. This is easier if you add flaps as in the third figure. You now have a model of a regular tetrahedron.

How do you make a model of a tetrahedron which is not a regular one? Cut any triangular region out of cardboard or heavy paper. Use this as the base of your model. Label its vertices A, B, and C. Cut out another triangle with one of its edges the same length as $\overline{AB}$. Now, with tape, fasten these two triangles together along edges of equal length. Use edge (AB) for this, for instance. Two of the vertices of the second triangle are now considered labeled A and B. Label the other vertex of the second triangle D. Cut out a third triangular region with one edge the length of $\overline{AD}$ and another the length of $\overline{AC}$. Do not make the angle between these edges too large or too small. Now, with tape, fasten these edges of the third triangle to $\overline{AD}$ and $\overline{AC}$ so that the three triangles fit together in space. The model you have constructed so far will look something like a pyramid-shaped drinking cup if you hold the vertex A at the bottom, as in the drawing below. Finally, cut out a triangular region which will just fit the top, fasten it to the top and you will have a tetrahedron.



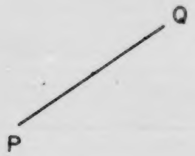
Exercises 6-1

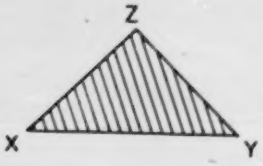
1. Make two cardboard or heavy paper models of a regular tetrahedron. Make your models so that their edges are each 3" long.
2. Make a model of a tetrahedron which is not regular.
3. In making the third face of a non-regular tetrahedron, what difficulties would you encounter if you made the angle DAC too large or too small?

6-2. Simplexes

A single point is probably the simplest object or set of points you can think of. A set consisting of two points is probably the next most simple set of points in space. But any two different points in space are on exactly one line, and are the endpoints of exactly one segment (which is a subset of the line). A segment has length but does not have width or thickness, so it does not have area. A segment or a line is one-dimensional. Either could be considered as the simplest one-dimensional object in space. In this chapter we will think about the segment, not the line.

A set consisting of three points is the next most simple set of points in space. If the three points are all on the same line, one is not much better off than if there were just two points. Let us agree, therefore, that our three points are not to be on the same line. Thus there is exactly one plane containing the three points and there is exactly one triangle with the three points as vertices. There is also exactly one triangular region which together with the triangle which bounds it, has the three points as vertices. This mathematical object, the triangle, together with its interior, is what we will think about. It is two-dimensional. It can be considered as the simplest two-dimensional object in space.





It seems rather clear that the next most simple set of points in space would be a set of four points. If the four points were all in one plane then the figure determined by the four points would apparently also be in one plane. It is important to require that four points are not all in any one plane. This requirement also guarantees that no three can be on a line. If any three were on a line, then there would be a plane containing that line and the fourth point and the four points would be in the same plane. Four points in space, not all in the same plane, suggest a tetrahedron. The four points in space are the vertices of exactly one solid tetrahedron. A solid tetrahedron is three-dimensional. It can be considered as the simplest three-dimensional object in space.

Each of the four objects described above may be thought of as the simplest of its kind. There are remarkable similarities among these objects. They all ought to have names that sound alike and remind us of their basic properties. Each of these is called a simplex. You can tell them apart by labeling each with its natural dimension. Thus a set consisting of a single point is called a 0-simplex. A segment is called a 1-simplex. A triangle together with its interior is called a 2-simplex. A solid tetrahedron (which includes its interior) is called a 3-simplex.

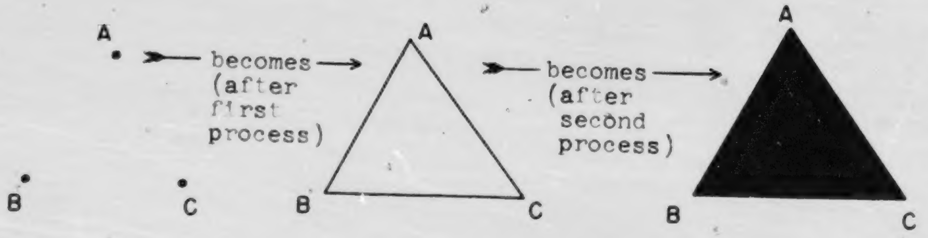
Let us make up a table to help us keep these ideas in order.

A set consisting of:	determines:	which is called a:
one point	one point (itself)	0-simplex
two points	a segment	1-simplex
three points not all on any one line	a triangle together with its interior	2-simplex
four points not all on any one plane	a solid tetrahedron	3-simplex

There is another way to think about the dimension of these sets. In this we think of the notion of betweenness, of a point being between two other points.

Let us start with two points. Consider these two points and all points between them. The set formed in this way is a segment. Now take the segment together with all points which are between any two points of the segment. We get just the same segment. No new points were obtained by "taking points between" again. The process of "taking points between" was used just once. We get a one-dimensional set, a 1-simplex.

Next consider three points not all on the same line. Then let us apply our process. We take these points together with all points which are between any two of them. At this stage we have a triangle but not its interior. We apply the process again by taking the set we already have (the triangle) together with all points which are between any two points of this set. We get the union of the triangle and its interior. If we apply the process again we don't get anything new. We need to use the process just twice. We get a two-dimensional set, a 2-simplex.



Next let us consider four points not all on the same plane. We apply the process of "taking points between" and we get the union of the edges of a tetrahedron. We apply the process again and get the union of the faces. We apply it once more and get the solid tetrahedron itself. We apply it again and still get just the solid tetrahedron. We need to use the process just three times. We get a three dimensional set, a 3-simplex.

If we had just one point, the application of the process would still leave us with just the one point. We need apply the process zero times. We get a zero-dimensional set, a 0-simplex. (We mention this case last because we have to understand the process before it can make much sense.)

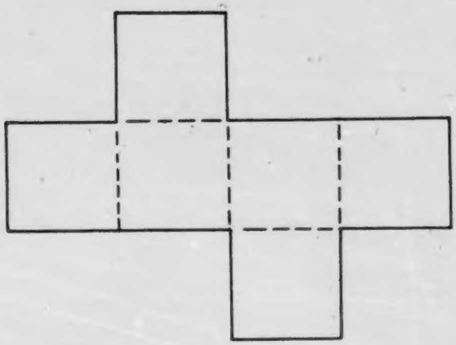
Let us consider a 3-simplex. Look at one of your models of a tetrahedron. It has four faces and each face is a 2-simplex. It has six edges and each edge is a 1-simplex. It has four vertices and each vertex is an 0-simplex.

Exercises 6-2

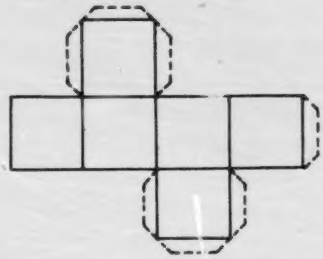
1. (a) A 2-simplex has how many 1-simplexes as edges?
   (b) It has how many 0-simplexes as vertices?
2. A 1-simplex has how many 0-simplexes as vertices?
3. Using models show how two 3-simplexes can have an intersection which is exactly one vertex of each.
4. Using models show how two 3-simplexes can have an intersection which is exactly one edge of each.

6-3. Models of Cubes

An ordinary box could be made from six rectangular faces. They have to fit and you have to put them together correctly. There is a rather easy way to make a model of a cube.

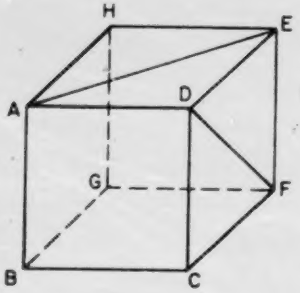


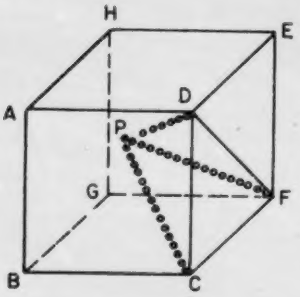
Draw six squares on heavy paper or cardboard as in the drawing above. Cut around the boundary of your figure and fold (or crease) along the dotted lines. Use cellulose tape or paste to fasten it together. If you are going to use paste it will be useful to have flaps as indicated in the drawing below.



Can the surface of a cube be regarded as the union of 2-simplexes (that is, of triangles together with their interiors)? Can a solid cube be regarded as the union of 3-simplexes (that is of solid tetrahedrons)? The answer to both of these questions is "yes".

Each face of a cube can be considered to be the union of two 2-simplexes. The drawing on the left below shows a cube with two of its faces subdivided into two 2-simplexes each. The face ADEH appears as the union of (ADE) and (AEH) for example. The other face which is indicated as subdivided is CDEF. It appears as the union of (CDF) and (DFE). The other faces have not been subdivided, but you can think of each of them as the union of two 2-simplexes. Thus the surface of the cube can be thought of as the union of twelve 2-simplexes.





With the surface regarded as the union of 2-simplexes you may regard the solid cube as the union of 3-simplexes (solid tetrahedrons) as follows. Let P be any point in the interior of the cube. For any 2-simplex on the surface, (CDF), for example, (PCDF) is a 3-simplex. In the figure on the right above, P is indicated as inside the cube. The 1-simplexes (PC), (PD), and (PF) are also inside the cube. Thus with twelve 2-simplexes on the surface, we would have twelve 3-simplexes whose union would be the cube. The solid cube is the union of 3-simplexes in this "nice" way.

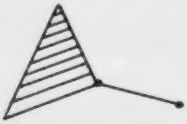
Exercises 6-3

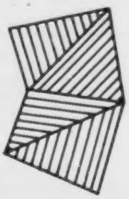
1. Make two models of cubes out of cardboard or heavy paper. Make them with each edge 2" long.
2. On one of your models, without adding any other vertices, draw segments to express the surface of the cube as a union of 2-simplexes. Label all the vertices on the model A, B, C, D, E, F, G, and H. Think of a point P in the interior of the cube. Using this point and the vertices of the 2-simplexes on the surface list the twelve 3-simplexes whose union is the solid cube.
3. On the same cube as in Problem 2, mark a point in the center of each face. (Each should be on one of the segments you drew in Problem 2.) Using as vertices the corners of the cube and these six new points you have marked, draw segments to indicate the surface of the cube as the union of 2-simplexes. The surface is expressed as the union of how many 2-simplexes in this new way?

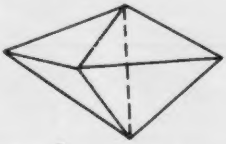
4. Think about a geometric figure formed by putting a square-based pyramid on each face of a cube. This is one example of a polyhedron. The surface of this polyhedron has how many triangular faces? Can you set up a one-to-one correspondence between this polyhedron, vertex for vertex, edge for edge, and 2-simplex for 2-simplex, and the surface of the cube subdivided into 2-simplexes as in Problem 3?

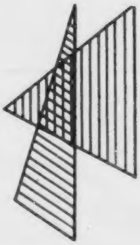
6-4. Polyhedrons

A polyhedron is defined as the union of a finite number of simplexes. It could be just one simplex, or maybe the union of seven simplexes, or maybe of 7,000,000 simplexes. What is being said is that it is the union of some particular number of simplexes. In the previous section, observe that a solid cube, for example, was the union of twelve 3-simplexes. The figures below represent the unions of simplexes.





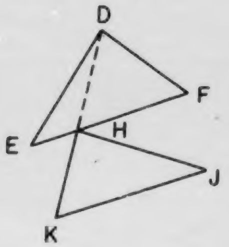


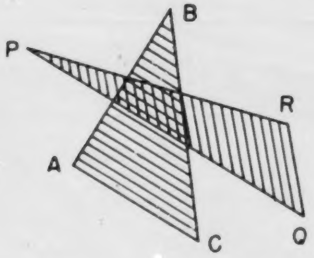


The figure on the left represents a union of a 1-simplex and a 2-simplex which does not contain the 1-simplex. It is therefore of mixed dimension. In what follows, you will not be concerned with polyhedrons (or polyhedra) of mixed dimension. Assume that a polyhedron is the union of simplexes of the same dimension. A 3-dimensional polyhedron will be spoken of as one which is the union of 3-simplexes. A 2-dimensional polyhedron is one which is the union of 2-simplexes. A 1-dimensional polyhedron is one which is the union of 1-simplexes. (Any finite set of points could be thought of as a 0-dimensional polyhedron but these will not be dealt with here.)

The figure on the right above represents a polyhedron which seems to be the union of two 2-simplexes (triangular regions) but they don't intersect nicely. It is more convenient to think of a polyhedron as the union of simplexes which intersect nicely as in the middle two figures. Just what is meant by simplexes *intersecting nicely*? There is an easy explanation for it. If *two simplexes of the same dimension intersect nicely, then the intersection must be a face, or an edge, or a vertex of each*.

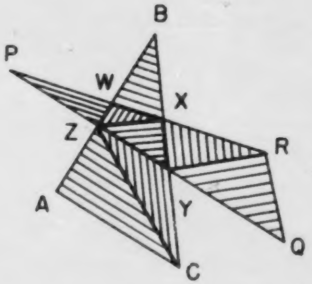
Let us look more closely at the union of simplexes which do not intersect nicely. In the figure on the right the 2-simplexes (DEF) and (HJK) have just the point H in common. They do not intersect *nicely*. While H is a vertex of (HJK), it is not of (DEF). However, the polyhedron which is the union of these two 2-simplexes is also the union of three 2-simplexes which do not intersect *nicely*, (DEH), (DHF), and (HJK).





The figure on the left represents the union of the 2-simplexes (ABC) and (PQR). They do not intersect nicely. Their intersection seems to be a quadrilateral together with its interior.

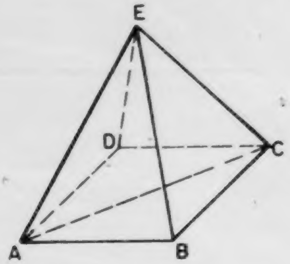
On the right it is indicated how the same set of points (the same polyhedron) can be considered to be a finite union of 2-simplexes which do not intersect nicely. The polyhedron is the union of the eight 2-simplexes, (ACZ), (CZY), (PZW), (XYZ), (WXZ), (BWX), (XYR), and (YQR).

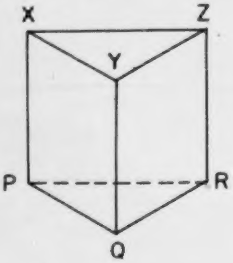


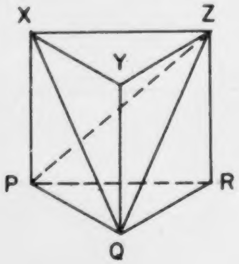
These examples suggest a fact about polyhedrons. If a polyhedron is the union of simplexes which intersect any way at all then the same set of points (the same polyhedron) is also the union of simplexes which intersect nicely. Except for the exercises at the end of this section, you shall always deal with unions of simplexes which intersect nicely. When convenient a polyhedron will be regarded as having associated with it a particular set of simplexes which intersect nicely and whose union it is. When you say the word "polyhedron," you may understand the simplexes to be there.

Is a solid cube a polyhedron, that is, is it a union of 3-simplexes? You have already seen that it is. Is a solid prism a polyhedron? Is a solid square-based pyramid? The answer to all of these questions is yes. In fact, any solid object each of whose faces is flat (that is, whose surface does not contain any curved portion) is a 3-dimensional polyhedron. It can be expressed as the union of 3-simplexes.

As examples let us look at a solid square-based pyramid and a prism with a triangular base.





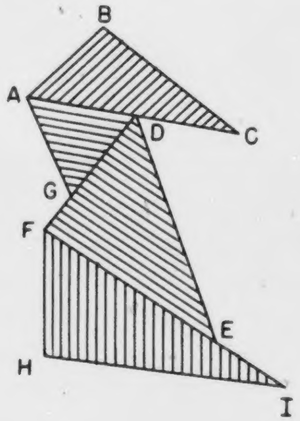


In the figure on the left the solid pyramid is the union of the two 3-simplexes (ABCE) and (ACDE). The figure in the middle represents a solid prism with a triangular base. The prism has three rectangular faces. Its bases are (PQR) and (XYZ). Here you can see how the solid prism may be expressed as the union of eight 3-simplexes. The same device is used that was used for the solid cube. First think about the surface as the union of 2-simplexes. You already have the bases as 2-simplexes. Then think of each rectangular face as the union of two 2-simplexes. In the figure on the right above, the face YZRQ is indicated as the union of (YZQ) and (QRZ), for instance. Now think about a point F in the interior of the prism. The 3-simplex (FQRZ) is one of eight 3-simplexes (each with F as a vertex) whose union is the solid prism. In the exercises you will be asked to name the other seven.

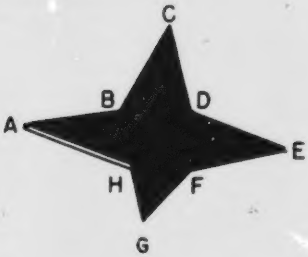
Finally, how do you express a solid prism with non-triangular bases as a 3-dimensional polyhedron (that is, as a union of 3-simplexes with nice intersections)? You can use a little trick. First express the base as a union of 2-simplexes and using these you may express the solid prism as a union of triangular solid prisms. And you may then express each triangular solid prism as the union of eight 3-simplexes. You can do this in such a way that all the simplexes intersect nicely.

Exercises 6-4

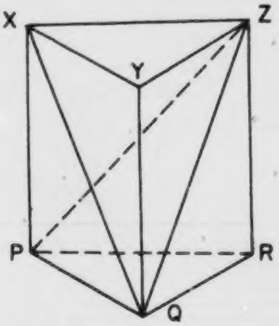
1. Draw two 2-simplexes whose intersection is one point and
   (a) the point is a vertex of each.
   (b) the point is a vertex of one but not of the other.
2. Draw three 2-simplexes which intersect nicely and whose union is itself a 2-simplex. (Hint: start with a 2-simplex as the union and subdivide it.)
3. You are asked to draw various 2-dimensional polyhedrons each as the union of six 2-simplexes. Draw one such that
   (a) No two of the 2-simplexes intersect.
   (b) There is one point common to all the 2-simplexes but no other point is common to any pair.
   (c) The polyhedron is a rectangle together with its interior.
4. The figure on the right represents a polyhedron as the union of 2-simplexes without nice intersections. Draw a similar figure yourself and then draw in three segments which will indicate the polyhedron as the union of 2-simplexes which intersect nicely.



5. The 2-dimensional figure on the right can be expressed as a union of 2-simplexes with nice intersections in many ways. Draw a similar figure and:



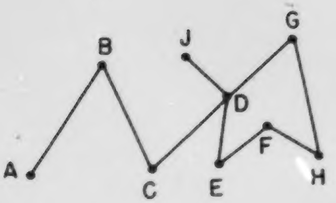
   (a) By drawing segments express it as the union of six 2-simplexes without using more vertices.
   (b) By adding one vertex near the middle (in another drawing of the figure), express the polyhedron as the union of eight 2-simplexes each having the point in the middle as one vertex.

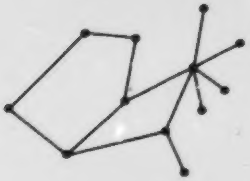


6. (a) List eight 2-simplexes whose union is the surface of the triangular prism on the right. (The figure is like that used earlier.)
   (b) Regarding F as a point in the interior of the prism list eight 3-simplexes (each containing F) whose union is the solid prism.
   (c) The figure shows the triangular prism (PQRXYZ) as the union of three 3-simplexes which intersect nicely. Name them.

6-5. One-Dimensional Polyhedrons

A 1-dimensional polyhedron is the union of a certain number of 1-simplexes (segments). A 1-dimensional polyhedron may be contained in a plane or it may not be. Look at a model of a tetrahedron. The union of the edges is a 1-dimensional polyhedron. It is the union of six 1-simplexes, and does not lie in a plane. You may think of the figure below as representing 1-dimensional polyhedrons that do lie in a plane (the plane of the page).

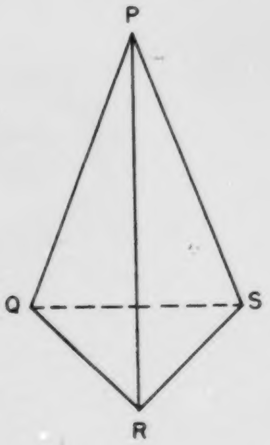




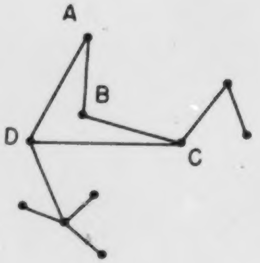
There are two types of 1-dimensional polyhedrons which are of special interest. A polygonal path is a 1-dimensional polyhedron in which the 1-simplexes can be considered to be arranged in order as follows. There is a first one and there is a last one. Each other 1-simplex of the polygonal path has one vertex in common with the 1-simplex which precedes it and one vertex in common with the 1-simplex which follows it. There are no extra intersections. The first and last vertices (points) of the polygonal path are called the endpoints.

Neither of the 1-dimensional polyhedrons in the figures above is a polygonal path. But each contains many polygonal paths. The union of (AB), (BC), (CD), (DG) and (GH) is a polygonal path from A to H. The union of (JD) and (DE) is a polygonal path from J to E and consists of just two 1-simplexes.

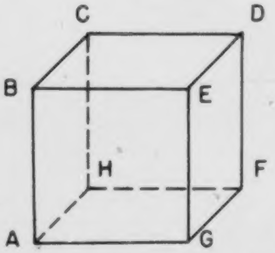
In the drawing of a tetrahedron on the right, the union of (PQ), (QR), and (RS) is a polygonal path from P to S (with endpoints P and S). The 1-simplex (PS) is itself a polygonal path from P to S. Consider the 1-dimensional polyhedron which is the union of the edges of the tetrahedron, and find another polygonal path from P to S in it. (Use a model if it helps you see it.) How many such polygonal paths are there from P to S?



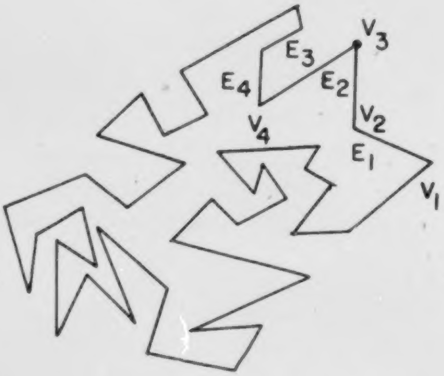
The union of two polygonal paths that have exactly their endpoints in common is called a simple closed polygon (it is also a simple closed curve). The 1-dimensional polyhedron on the right is not a simple closed polygon. But it contains exactly one simple closed polygon, namely the union of the polygonal paths ABC and ADC which have endpoints A and C in common.



The union of the edges of the cube in the drawing is a 1-dimensional polyhedron. It contains many simple closed polygons. One is the union of (AB), (BE), (EG), and (GA). Another is the union of (AB), (BC), (CD), (DE), (EG), and (GA). List the vertices naming at least two more simple closed polygons containing (AB) and (GA). (Use a model if it helps you see it.)

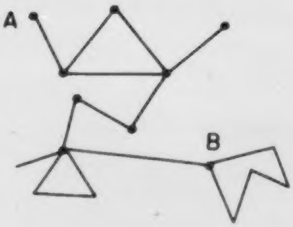


There is one very easy relationship on any simple closed polygon. The number of 1-simplexes (edges) is equal to the number of vertices. Consider the figure on the right. Suppose you start at some vertex. Then take an edge containing this vertex. Next take the other vertex contained in this edge and then the other edge containing this second vertex. You may think of numbering the vertices and edges as in the figure. Continue the process. You will finish with the other edge which contains our original vertex. You will start with a vertex and finish with an edge after having alternated vertices and edges as we go along. Thus the number of vertices is the same as the number of edges.

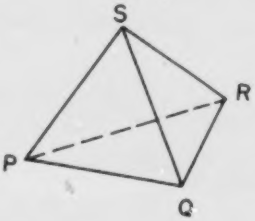


Exercises 6-5

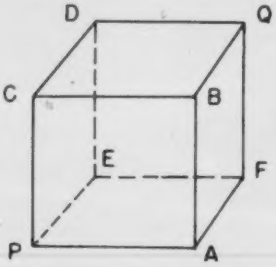
1. The figure on the right represents a 1-dimensional polyhedron. How many polygonal paths does it contain with endpoints A and B? How many simple closed polygons does it contain?



2. (a) The union of the edges of a 3-simplex (solid tetrahedron) contains how many simple closed polygons?
   (b) Name them all.
   (c) Name one that is not contained in a plane.
   (Use a model if you wish.)



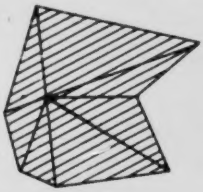
3. Let P and Q be vertices of a cube which are diametrically opposite each other (lower front left and upper back right). Name three polygonal paths from P to Q each of which contains all the vertices of the cube and is in the union of the edges. (Use a model if you wish.)



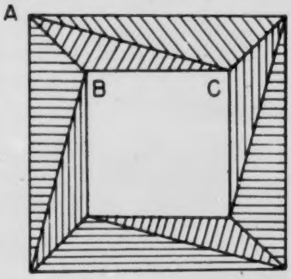
4. Draw a 1-dimensional polyhedron which is the union of seven 1-simplexes and contains no polygonal path consisting of more than two of these simplexes.
5. On the surface of one of your models of a cube, draw a simple closed polygon which intersects every face and which does not contain any of the vertices of the cube.

6-6. Two-Dimensional Polyhedrons

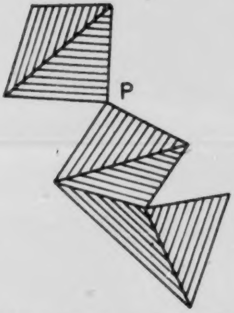
A 2-dimensional polyhedron is a union of 2-simplexes. As stated before, it is agreed that the 2-simplexes are to intersect nicely. That is, if two 2-simplexes intersect, then the intersection is either an edge of both, or a vertex of both. There are many 2-dimensional polyhedrons; some are in one plane but many are not in any one plane. The surface of a tetrahedron, for instance, is not in any one plane. Let us first consider a few 2-dimensional polyhedrons in a plane. In drawing 2-simplexes in a plane their interiors will be shaded.



Every 2-dimensional polyhedron in a plane has a boundary in that plane. The boundary is itself a 1-dimensional polyhedron. The boundary may be a simple closed polygon as in the figure on the right. In the figure on the right below a polyhedron has been indicated as the union of eight 2-simplexes. (ABC) is one of them. The boundary is the union of two simple closed polygons, the inner square and the outer square. These two polygons do not intersect.



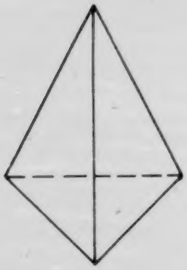
The figure on the right represents a 2-dimensional polyhedron which is the union of six 2-simplexes. The boundary of this polyhedron in the plane is the union of two simple closed polygons which have exactly one vertex of each in common, the point P.

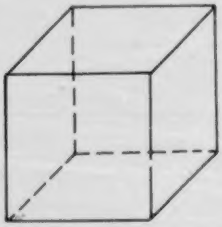


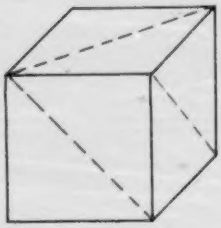
Suppose a 2-dimensional polyhedron in the plane has a boundary which is a simple closed polygon (and nothing else). Then the number of 1-simplexes (edges) of the boundary is equal to the number of 0-simplexes (vertices) of the boundary. You have already seen, in the previous section, why this must be true.

There are many 2-dimensional polyhedrons which are not in any one plane. The surface of a tetrahedron is such a polyhedron. The surface of a cube is another. You have seen that the surface of a cube may be considered to be expressed as a union of 2-simplexes.

These are examples of some 2-dimensional polyhedrons which are themselves surfaces or boundaries of 3-dimensional polyhedrons. Let us consider these two surfaces, the surface of a tetrahedron and the surface of a cube.







You may look at the drawings above or you may look at some models (or both). As you have observed earlier, the surface of a cube may be considered in several different ways. You may think of the faces as being square regions (as in the middle figure above) or you may think of each square face as subdivided into two 2-simplexes (as suggested in the figure on the right above). In any case, it seems clear what is meant by faces, edges, or vertices. Let us count the numbers of such objects for a given polyhedron. The letter F will be used for the number of faces, E for the number of edges and V for the number of vertices. If you are counting from models and do not observe patterns to help you count, it is usually easier to check things off as you go along. That is, mark the objects as you count them.

Let us make up a table of our results.

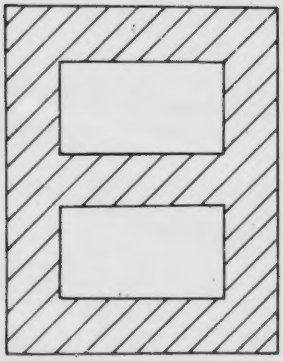
	F	E	V
Surface of tetrahedron	?	6	?
Surface of cube (square faces)	?	?	8
Surface of cube (two 2-simplexes on each square face)	12	?	?

It is not easy by considering just these special polyhedrons to observe any relationship among the numbers you get. What you are looking for is a relationship, if any, which will be true not only for any one of these 2-dimensional polyhedrons but also for any others like these. See if you can discover a relationship which is true in each case.

Exercises 6-6

1. Make up a table as in the text showing F, V, and E for the 2-dimensional polyhedrons mentioned there.
2. (a) Draw a 2-dimensional polyhedron in the plane with the polyhedron the union of ten 2-simplexes such that its boundary is a simple closed polygon.
   (b) Similarly draw another such polyhedron such that its boundary is the union of three simple closed polygons having exactly one point in common.
   (c) Draw another, such that its boundary is the union of two simple closed polygons which do not intersect.
3. Draw a 2-dimensional polyhedron in the plane with the number of edges in the boundary
   (a) equal to the number of vertices,
   (b) one more than the number of vertices,
   (c) two more than the number of vertices.
4. Draw a 2-dimensional polyhedron which is the union of three 2-simplexes with each pair having exactly an edge in common. Do you think that there exists in the plane a polyhedron which is the union of four 2-simplexes such that each pair has exactly an edge in common? Does one exist in space?
5. On one of your models of a cube, mark six points one at the center of each face. Consider each face to be subdivided into four 2-simplexes each having the center point as a vertex. Count F (the number of 2-simplexes), E (the number of 1-simplexes), and V (the number of 0-simplexes) for this subdivision of the whole surface. Keep your answers for later use.
6. Do Problem 5 without using a model and without doing any actual counting. Just figure out how many of each there must be. For instance, there must be 14 vertices, 8 original ones and 6 added ones.

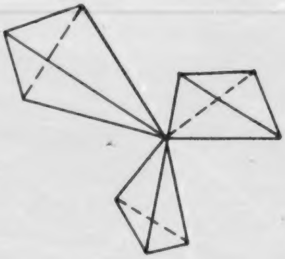
7. Express the polyhedron on the right as a union of 2-simplexes which intersect nicely (in edges or vertices of each other).



6-7. Three-Dimensional Polyhedrons

A 3-simplex is one 3-dimensional polyhedron. A solid cube is another 3-dimensional polyhedron. Any union of 3-simplexes is a 3-dimensional polyhedron. It will be assumed again that the simplexes of a polyhedron intersect nicely. That is, that if two 3-simplexes intersect, the intersection is a 2-dimensional face (2-simplex) of each or an edge (1-simplex) of each or a vertex (0-simplex) of each.

Any 3-dimensional polyhedron has a surface (or boundary) in space. This surface is itself a 2-dimensional polyhedron. It is the union of several 2-simplexes (which intersect nicely). The surface of the 3-dimensional polyhedron represented by the drawing on the right consists of the surfaces of three tetrahedrons which have exactly one point in common.



The simplest kinds of surfaces of 3-dimensional polyhedrons are called simple surfaces. The surface of a cube and the surface of a 3-simplex are both simple surfaces. There are many others. Any surface of a 3-dimensional polyhedron obtained as follows will be a simple surface. Start with a solid tetrahedron. Then fasten another to it so that the intersection of the one you are adding with what you already have is a face of the one you are adding. You may keep adding more solid tetrahedrons in any combination or of any size provided that each one you add in turn intersects what you already have in a set which is exactly a union of one, two or three faces of the 3-simplex you are adding. The surface of any polyhedron formed in this way will be a simple surface.

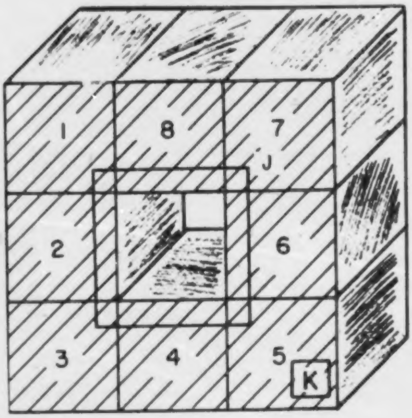
Class activity. Take five models of regular tetrahedrons of edges 3". Put a mark on each of the four faces of one of these. Now fasten each of the others in turn to one of the marked faces. The marked one should be in the middle and you won't see it any more. The surface of the object you have represents a simple surface. You can see how to fasten a few more tetrahedrons to get more and more peculiar looking objects. Suppose it is true that whenever you add a solid tetrahedron the intersection of what you add with what you already have is one face, two faces, or three faces of the one you add. The surface of what you get will be a simple surface.

One can also fasten solid cubes together to get various 3-dimensional polyhedrons. If you wish them to have simple surfaces, you must follow a rule like the one given before. The cubes must be fastened together in such a way that the intersection of the polyhedron you already have with the cube you are adding is a set which is bounded on each surface by a simple closed polygon. For example, the intersection might be a face or the union of two or more adjacent faces of the cube you are adding.

Finally an interesting property of simple surfaces is mentioned. Draw any simple closed polygon on a simple surface. Then this polygon separates the simple surface into exactly two sets each of which is connected, i.e., is in one piece.

Class activity. On the surface of one of the peculiar 3-dimensional polyhedrons (with simple surface) that you have constructed above, let one student draw any simple closed polygon (the wilder the better). It need not be just one face. Then let another student start coloring somewhere on the surface but away from the polygon. Let him color as much as is possible without crossing the polygon. Then let another student start coloring with another color at any previously uncolored place. Color as much as possible but do not cross the polygon. When the second student has colored as much as possible, the whole surface should be colored.

If you don't carefully follow the instructions for constructing a polyhedron with a simple surface, you may get a polyhedron whose surface is not simple. Suppose, for instance, you fasten eight cubes together as in the drawing shown below. The polyhedron looks something like a square doughnut. Note that in fitting the eighth one, the intersection of the one you are adding with what you already have is the union of two faces which are not adjacent. The boundary (on the eighth cube) of the intersection is two simple closed polygons not just one as it should be. There are many simple closed polygons on this surface which do not separate it at all.



The polygon J does not separate it. The polygon K does.

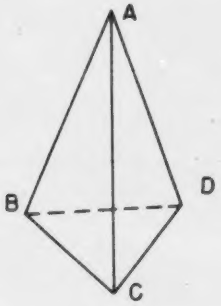
Exercises 6-7

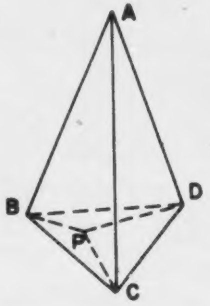
1. Using a block of wood (with corners sawed off if possible), draw a simple closed polygon on the surface making it intersect most or all of the faces of the solid. Start coloring at some point. Do not cross the polygon. Color as much as you can without crossing the polygon. When you have colored as much as you can, start coloring with a different color on some uncolored portion. Again color as much as you can without crossing the polygon. You should have the whole surface colored when you finish.
2. Go through the same procedure as in Problem 1 but with another 3-dimensional solid with simple surface. Use one of your

models or another block of wood. Make your simple closed polygon as complicated as you wish.

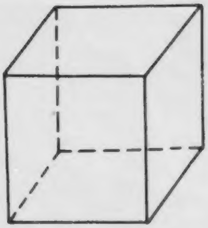
6-8. Counting Vertices, Edges, and Faces - The Euler Formula

In Section 6-6 you were asked to do some counting. Look at the problem in another way. A few of you may have discovered a relationship between F, E, and V. Consider the tetrahedron in the figure below. Its surface is a simple surface. What relationship can you find among the vertices, edges, and faces of it?





There are the same number of edges and faces coming into the point A, three of each. One may see that on the base there are the same number of vertices as edges. You have two objects left over: the vertex A at the top and the face (BCD) at the bottom. Otherwise you have matched all the edges with vertices and faces. So $V + F - E = 2$. Now let us ask what would be the relationship if one of the faces or the base were broken up into several 2 simplexes. Suppose you had the base broken up into three 2-simplexes by adding one vertex P in the interior of the base. The figure on the right above illustrates this. The counting would be the same until you got to the base and you would be able to match the three new 1-simplexes which contain P with the three new 2-simplexes on the base. You have lost the face which is the base but you have picked up one new vertex P. Thus the number of vertices plus the number of 2-simplexes is again two more than the number of 1-simplexes. $V + F - E = 2$.



Next let us look at a cube. There is a drawing of one on the right. The cube has how many faces? How many edges? How many vertices? Is the sum of the number of vertices and the number of faces two more than the number of edges? Let us see why this must be.

(1) The number of vertices on the top face is the number of edges on the top face.

(2) The number of vertices on the bottom face is the number of edges on the bottom face.

(3) The number of vertical faces is the number of vertical edges.

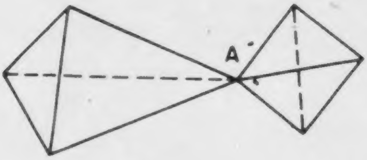
(4) All the vertices and edges are now used up. All the vertical faces are now used up. The top and bottom faces are left.

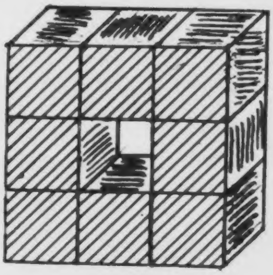
So $V + F = E + 2$ or $F + V - E = 2$.

What would happen if each face were broken up into two 2-simplexes? For each face of the cube you would now have two 2-simplexes. But for each face you would have one new 1-simplex lying in it. Other things are not changed. Hence $V + F - E$ is again 2.

Suppose you have any simple surface. Then do you suppose that $V + F - E = 2$? In the exercises you will be asked to verify this formula, which is known as the Euler Formula, in several other examples. Euler (pronounced "Oiler") is the name of a famous mathematician of the early 18th century.

Let us now observe that the formula does not hold in general for surfaces which are not simple. Consider the two examples below.



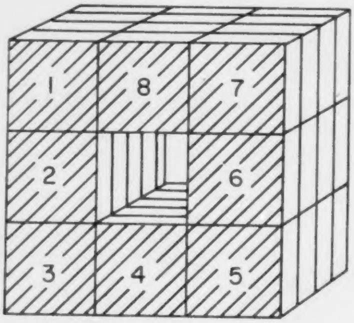


In the figure on the left (the union of the two tetrahedrons having exactly the vertex A in common), what is V + f - E ? Count and see. Use models of two tetrahedrons if you wish. V + F - E should be 3. You may reason this way. On each tetrahedron separately the number of faces plus the number of vertices is the number of edges plus 2. But the vertex A would have been counted twice. So V + F is one less than E + 4.

The figure on the right above is supposed to represent the union of eight solid cubes as in the last section. The possible ninth one in the center is missing. Count all the faces (of cubes), edges and vertices which are in the surface. For this figure V + F - E should be 0. As a starter, V should be 32.)

Exercises 6-8

1. Take a cardboard model of a non-regular tetrahedron. In each face add a vertex near the middle. Consider the face as the union of three 2-simplexes so formed. Give the count of the faces, edges, and vertices of the 2-simplexes on the surface. How do the faces, edges, and vertices of this polyhedron compare with those of the polyhedron you get by attaching four regular tetrahedrons to the four faces of a fifth? Does the Euler Formula hold for such a polyhedron?
2. Take a model of a cube. Subdivide it as follows. Add one vertex in the middle of each edge. Add one vertex in the middle of each face. Join the new vertex in the middle of each face with the eight other vertices now on that face. You should have eight 2-simplexes on each face. Compute F, V, and E. Do you get V + F - E = 2?
3. Consider the squarish doughnut which is the union of 8 cubes. Without cube number 8, the surface is simple and V + F - E = 2. What is the effect of adding cube 8? You lose two faces (one on cube 1 and one on cube 7). You gain 4 faces and 4 edges on cube 8 and that is all. Hence for the whole polyhedron, V + F - E = ?

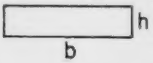


Chapter 7

VOLUMES AND SURFACE AREAS

7-1. Areas of Plane Figures

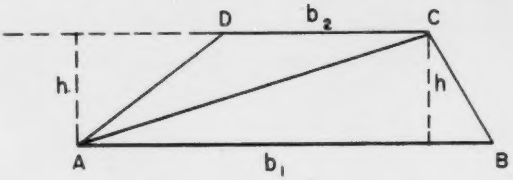
You recall that area of a surface is the number of square units contained in it. The area of a rectangle means the area of the rectangular closed region. The area of a rectangle is the product of the measures of the length and width.
Stated as a formula, A▭ = bh .
A symbol such as the rectangle in "A▭" will sometimes be used.



For a square, $A_{\square} = s \cdot s$ or, $A_{\square} = s^2$.

A parallelogram has the same area as a rectangle of the same height and base. Stated as a formula, A▱ = bh .

A trapezoid is a quadrilateral, only two of whose sides are parallel.

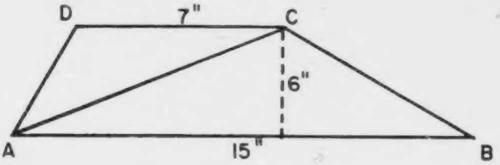


If a diagonal (such as $\overline{AC}$) is drawn, the interior of the trapezoid is separated into two triangular regions. Note that the altitudes of both triangles are congruent, but the bases b_1 and b_2, of the two triangles have different measures. The area of the trapezoid is the sum of the areas of the two triangles:

Area of ABCD = Area of ABC + Area of ADC

$$ABCD = \frac{1}{2}hb_1 + \frac{1}{2}hb_2 \ .$$

Example: In the trapezoid ABCD find the area by finding the sum of the areas of the two triangles into which the diagonal $\overline{AC}$ separates it.



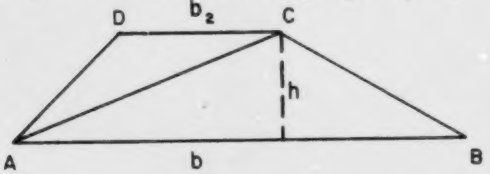
$$A_{\Delta ABC} = \frac{1}{2} \cdot 6 \cdot 15 = 45$$

$$A_{\Delta ADC} = \frac{1}{2} \cdot 6 \cdot 7 = 21$$

$$= 45 + 21 = 66$$

The area of the trapezoid is 66 square inches.

The above method of finding the area of a trapezoid can be simplified by using the distributive property.



$$A_{ABCD} = \frac{1}{2} h \cdot b_1 + \frac{1}{2} h \cdot b_2$$

Then by the distributive property,

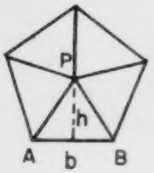
$$A_{ABCD} = \frac{1}{2} h (b_1 + b_2)$$

The formula may also be written:

$$A_{ABCD} = \frac{h}{2}(b_1 + b_2) \quad \text{or} \quad h \frac{(b_1 + b_2)}{2} .$$

The area of a trapezoid can be obtained by multiplying the measure of the height by the average of the measures of the bases.

Area of a Regular Polygon.



Recall that a regular polygon is defined to be a polygon whose sides have equal measures and whose angles have equal measures.

Join the center of the regular polygon to each vertex of the polygon. (The center is the point in the interior which is equally distant from the vertices of the polygon.) If there are n vertices there will be n congruent triangles.

The area of any such regular polygon will be the sum of the areas of the triangles. In the figure,

$$A\triangle = \frac{1}{2}hb$$

There are five such triangles, so:

$$A⬠ = \frac{1}{2}hb + \frac{1}{2}hb + \frac{1}{2}hb + \frac{1}{2}hb + \frac{1}{2}hb$$

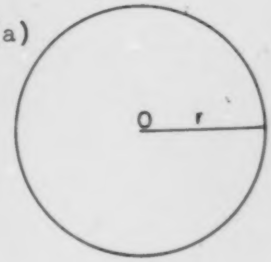
$$= \frac{1}{2}h\ (b + b + b + b + b)$$

But $(b + b + b + b + b)$ is the measure of the perimeter of the polygon. Thus $A⬠ = \frac{1}{2}hp$.

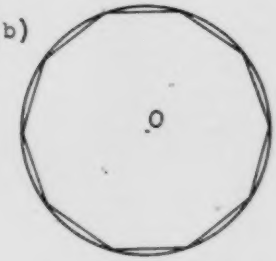
Area of a Circle

Consider circle O in the figures below.

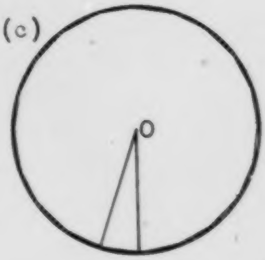
(a)



(b)



(c)

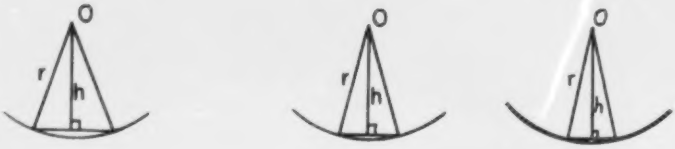


A regular polygon of n sides is inscribed in a circular region if the vertices of the regular polygon are points on the circle. It is clear from the figures above that the more sides the inscribed polygon has, the shorter will be the length of each side. As n gets larger and larger, it will be more and more difficult to distinguish between the regular polygon and the circle. You could say that the area of the interior of the inscribed polygon is approximately equal to the area of the interior of the circle, but there is always some portion of the area of the circle which is not contained in the interior of an inscribed regular polygon. However, for large values of n, the areas are very close, and you can think of the area of the interior of the circle as the upper limit of that of the inscribed regular polygons.

It should also be noted that the larger n (the number of sides) becomes, the closer the measure of the distance from the center of the inscribed polygon to a side gets to the measure of the radius of the circle. You have seen that the number of square units of area in the polygon is $\frac{1}{2}hp$, but you have just observed that when n gets very large, h gets very close to r, and p gets close to $2\pi r$ so you are led to conclude:

If r is the number of linear units in the radius of a circle, and A is the number of square units of area in its interior, then,

$$A = \frac{1}{2}r(2\pi r)$$
$$A = \pi r^2$$



Exercises 7-1

Find the area of each of the following figures:

	Figure	Measurements
1.	Rectangle ABCD	$\overline{AB}$ is 4" long, $\overline{BC}$ is 12" long.
2.	Square XYZW	YZ is $3\frac{1}{2}$ ft. long
3.	Parallelogram ABCD	$\overline{AB}$ is 16 in. long; the height is 15 in.
4.	Triangle XYZ	The base $\overline{YZ}$ is 38 ft. long, the height $\overline{XW}$ is 37 ft.
5.	A circle	Length of radius is 4.5 in. ($\pi \approx 3.14$)

6. Find the area of trapezoids:

Height	Upper Base	Lower Base
8 in.	6 in.	13 in.
5.4 ft.	9.8 ft.	12.7 ft.
$2\frac{1}{2}$ ft.	$3\frac{1}{4}$ ft.	$6\frac{1}{4}$ ft.

7. The area of a trapezoid is 696 sq. in. The lengths of the bases are 23 in. and 35 in. Find the height of the trapezoid.

8. Compute the area of the interior of each of the following circles. The measurement in each case are in inches. Express the answer in terms of π.

(a) $r = 5$ (c) $r = 20$

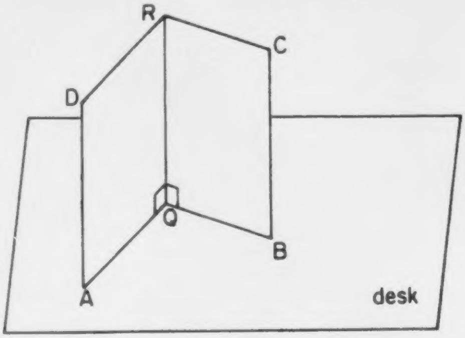
(b) $r = 10$ (d) $r = 4\frac{1}{2}$

9. Find the areas of the following regular polygons:

	Kind of Polygon	Distance from the Center to a Side	Length of a Side
(a)	Hexagon	17.3 inches	20 inches
(b)	Pentagon	27.5 inches	40 inches
(c)	Octagon	72.5 feet	60 feet
(d)	Decagon	30.8 inches	20 inches

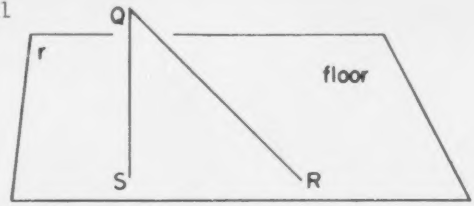
7-2. Planes and Lines

Take a piece of notebook paper as shown, first folding it so that $\overline{AD}$ falls on $\overline{BC}$. Now take the paper and set it on your desk as shown, in the position of a partly opened book, so that segments $\overline{AQ}$ and $\overline{BQ}$ lie on the plane of the desk top. Would you agree that $\overline{QR}$ is now perpendicular to the desk top? If so, notice that you have found a line perpendicular to a plane by making it perpendicular to just two different lines in the plane. This illustrates the following property of perpendiculars.



Property 1. If a line is perpendicular to two distinct intersecting lines in a plane, it is perpendicular to the plane.

From a consideration of several segments from a point Q to a plane, such as $\overline{QR}$ and $\overline{QS}$, we state



Property 2. The shortest segment from a point Q outside a plane r to the plane r is the segment perpendicular to that plane.

This shortest distance is called the distance from Q to r.

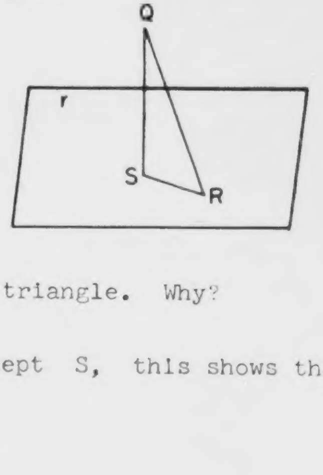
Extending the idea we have about the distance between two parallel lines to two parallel planes, we have:

Property 3. If two planes are parallel the (perpendicular) distances from different points of one plane to the other plane are all the same.

The constant distance in Property 3 is called the distance between the parallel planes. Actually the segments involved in Property 3 are perpendicular to both planes, as is true for the lateral edges of a right prism.

Exercises 7-2

1. Give five examples of pairs of parallel planes with lines perpendicular to both planes in each example.
2. You actually could have proved Property 2 instead of observing it by experiment.



Let S be the point of r so that $\overline{QS}$ is perpendicular to r. Draw segment $\overline{SR}$.

(a) $\angle$ QSR is a right angle. Why?

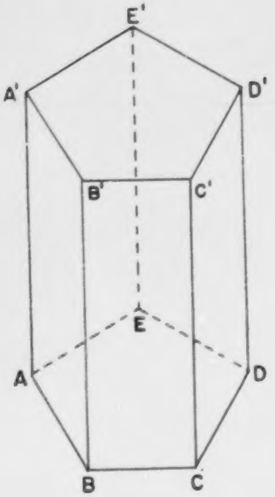
(b) $\overline{QR}$ is the hypotenuse of a right triangle. Why?

(c) $\overline{QR}$ is longer than $\overline{QS}$. Why?

But since R was any point of r except S, this shows that $\overline{QS}$ is the shortest segment.

7-3. Right Prism

In general, a right prism is a figure obtained from two congruent polygons, so located in parallel planes that when the segments are drawn joining corresponding vertices of the polygons,

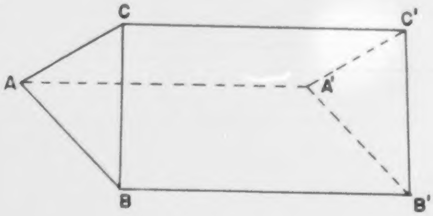


the quadrilaterals obtained are all rectangles. The prism is the union of the closed rectangular regions and the two closed polygonal regions. The rectangular regions are called the faces of the prism, the segments are its edges, and the points where two or more edges meet are called vertices. The bases of the prism are the regions bounded by the original polygons.

As was indicated, the figures described here are called right prisms. Later you will meet more general prisms for which the quadrilaterals mentioned are allowed to be any parallelograms rather than necessarily rectangles.

Rectangular right prisms are simply the right prisms whose bases are rectangular regions. A rectangular prism can be thought of as a prism in three different ways, since any pair of opposite faces can be used as bases. No other figure can be thought of as a right prism in more than one way.

In the drawings of the prisms it is often convenient to show the planes of the bases as horizontal. However, there should be no difficulty in identifying such figures when they occur in different positions. For example, the figure below represents a triangular right prism with bases ABC and A'B'C', even though it is shown resting on one of its rectangular faces.



Previously, the volume of a rectangular prism was discussed. This discussion showed that if the area of the base were 12 square units, then by using a total of 12 unit cubes of volume (some of which may be subdivided) you could form a layer one unit thick across the bottom of the prism. If the prism were $3\frac{1}{2}$ units high, it would take $3\frac{1}{2}$ such layers to fill the prism, or a total of $(12)(3\frac{1}{2})$ unit cubes, so the volume is 42 cubic units.

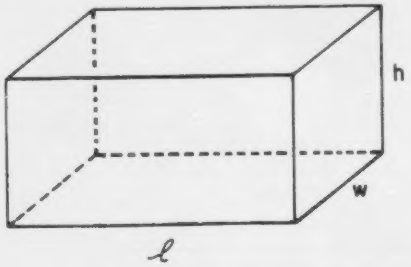
The interior of any right prism can be considered as consisting of a series of layers piled on each other. Thus, the following conclusion can be obtained:

> The number of cubic units of volume of a right prism is the product of the number of square units of area in the base and the number of linear units in the height.

In this statement, the term height means the perpendicular distance between the planes of the bases. It is the length of the segments from any vertex of one base to the corresponding vertex of the other base. Notice especially that the height is not measured vertically unless the planes of the bases happen to be horizontal. In the last figure, for example, the height is the length of any one of the segments $\overline{AA'}$, $\overline{BB'}$, or $\overline{CC'}$.

Rectangular Right Prisms

One right prism with which you are rather familiar is the rectangular right prism. A good example is a cereal box. The figure below represents such a prism.



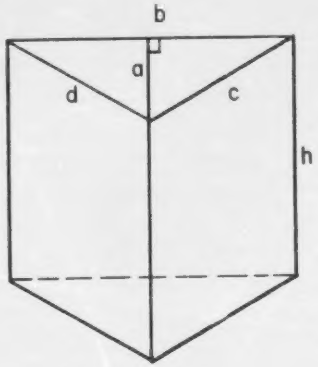
The measures of its length, width, and height will be represented by ℓ, w and h, respectively. Furthermore, S will represent the measure of its surface area and V will represent the measure of its volume. Recall the following formulas:

$S = 2(\ell w + \ell h + wh)$

$V = \ell wh$ or $V = Bh$, where B is the measure of the area of the base ($B = \ell w$). You will notice that in a prism of this type all faces are rectangles, so any pair of parallel faces can be considered as the bases.

A cube is a special case of the rectangular right prism in which all of its edges are congruent. Let us designate the measure of its edges by s. The formula for the surface area of a cube is $S = 6s^2$. The formula for the volume of a cube is $V = s^3$.

Triangular Right Prism



Since the area of the surface is the sum of the areas of the bases and the faces,

$$S = 2(\tfrac{1}{2}ab) + (hb + hc + hd)$$
$$S = ab + h(b + c + d)$$
$$S = ab + hp,$$

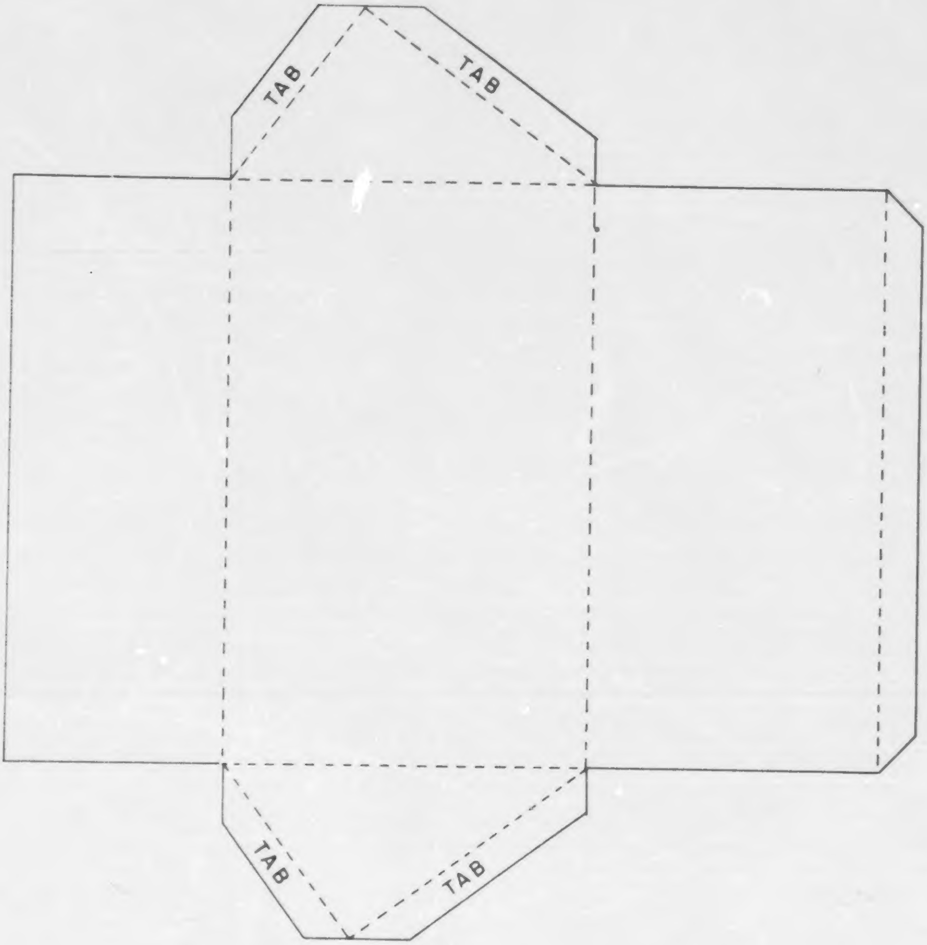
where p is the measure of the perimeter of the triangular base.

Also,

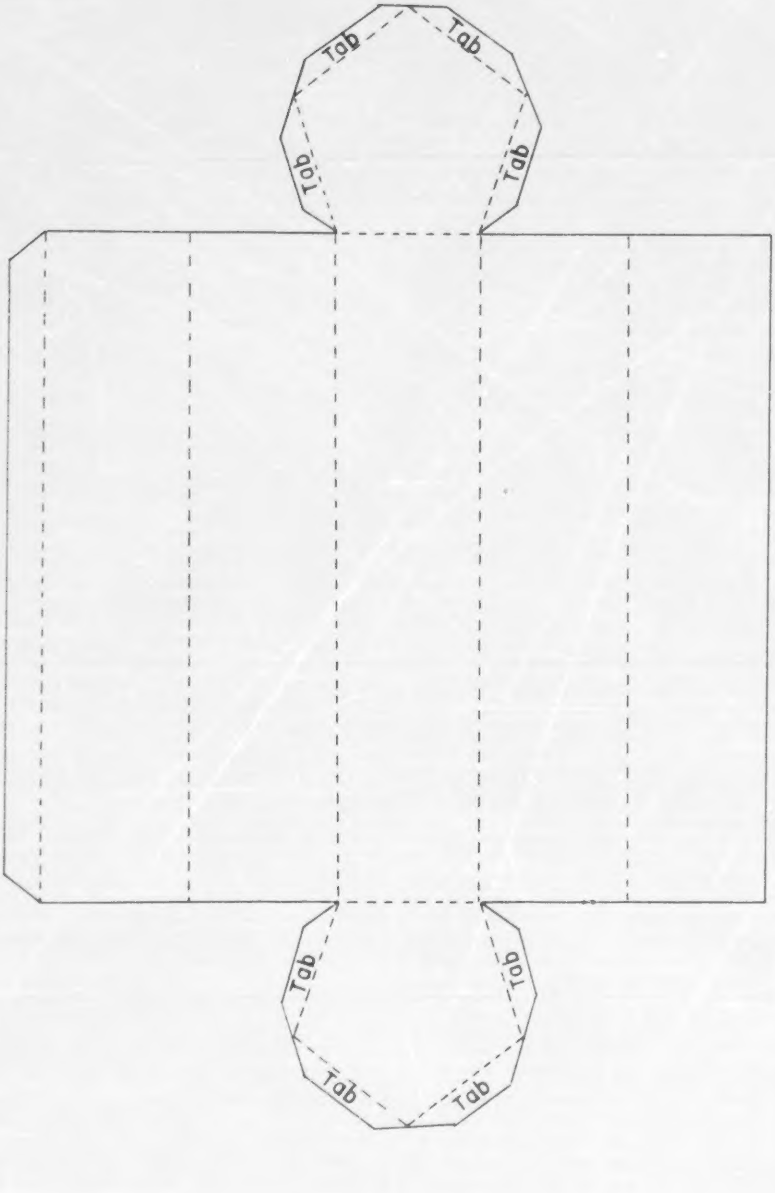
$$V = \tfrac{1}{2}abh \quad \text{or} \quad V = Bh,$$

where B is the measure of the area of the base.

Pattern for Triangular Prism



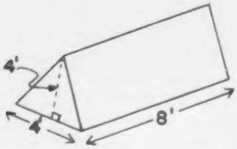
Pattern for Pentagonal Prism



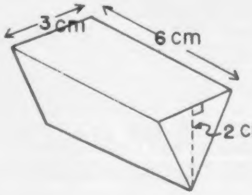
Exercises 7-3

1. Find the number of cubic units of volume for each of the prisms shown below:

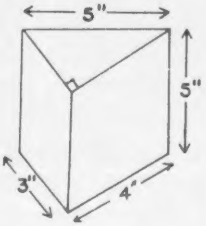
(a)



(b)

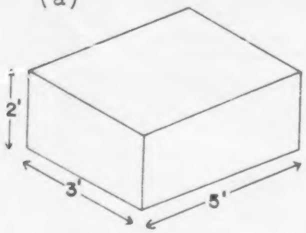


(c)

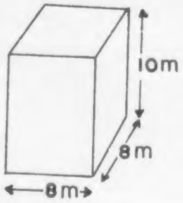


2. Find the number of cubic units of volume for each of the prisms shown below.

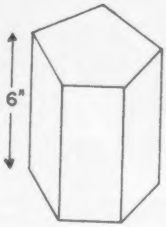
(a)



(b)



(c)



Area of the Pentagon is 21 square inches.

3. If B stands for the number of square units of area in the base of a prism, and h is the number of linear units in its height, write a sentence showing how to find the number V of cubic units of volume in the prism.
4. A container in the shape of a prism is 11 inches high and holds one gallon. How many square inches are there in the base? Do you know the shape of the base? (A gallon contains 231 cu. in.)
5. A triangular prism has a base which is a right triangle with the two perpendicular sides 3 inches and 6 inches. If the prism is 20 inches high, what is the volume in cubic inches?
6. How many edges, faces, and vertices are there on a triangular prism? a pentagonal prism? a hexagonal prism (6 sides)? an octagonal prism (8 sides)?

7. Compute the volume of each right rectangular prism whose measures are as follows:
   (a) $\ell = 1$, $w = 2$, $h = 2$
   (b) $\ell = 2\frac{1}{2}$, $w = 2$, $h = 2$

8. Calculate the surface area of each right rectangular prism of Problem 7.

9. Find the surface area and volume for Model 5, at the end of the chapter.

10. Construct Model 4. Find the area of its surface and its volume.

11. (a) Refer to patterns for Models 4 and 6 and find the perimeters of the bases.
    (b) Are these perimeters equal to each other?
    (c) Compare the volumes of these two models.
    (d, Are the volumes equal?

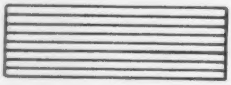
12. (a) When you computed the volumes of Models 4 and 5, did you find them equal?
    (b) Check (a) by filling one with salt and pouring it into the other.
    (c) Find the perimeters of the bases of these models. Are the perimeters equal?

7-4. Oblique Prisms

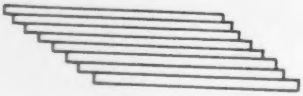
Consider two congruent polygons. Imagine them so placed in parallel planes that when the segments are drawn, joining corresponding vertices of the polygons, the quadrilaterals formed are all parallelograms, of which, at least two must be non-rectangular. This means that the lateral edges are not perpendicular to the bases. These parallelograms and original polygons determine closed regions. The union of these closed regions is called an *oblique prism*. The segments are its *edges*, and the points where two or more edges meet are vertices. The closed regions formed by the parallelograms are called *lateral faces* (or faces). The original polygonal closed regions are called *bases*. The segments joining corresponding vertices of the two bases are called *lateral edges*. (See Model 7).

As in the study of the right prisms we are interested in finding the surface area and volume of oblique prisms. There is no problem in finding the surface area, since this is obtained by finding the sum of the areas of the bases and lateral faces. Of course, in finding the areas of the lateral faces, you will be finding the areas of the parallelograms rather than rectangles.

Now let us consider the volume of an oblique prism. Take a stack of rectangular cards which are congruent. You may make such a stack or use a deck of playing cards. When you have the cards stacked so that all adjacent cards fit exactly you have an illustration of a right prism similar to the following illustration in cross section.



Now push the cards a bit so that the deck will have the following appearance in cross section.



You now have an illustration of an oblique prism. Of course it is not a perfect prism due to the thickness of the cards. The cards no longer fit smoothly.

Next let us consider a similarity between the two stacks illustrated above. You note that the bases are congruent and the distances between the bases are equal. In view of this discussion it would seem that we have the basis for making the following conjecture.

Conjecture: If two prisms have congruent bases and equal heights, they have equal volumes.

Since the conjecture seems to be borne out in practice, we will list it now as a property.

Property 4. If two prisms have congruent bases and equal heights, they have equal volumes.

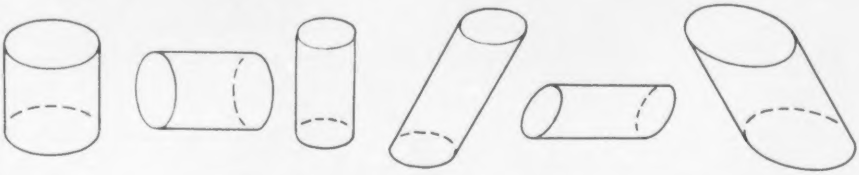
From Property 4 the volume of any prism is the same as that of a right prism whose base is congruent to the given one and having the same height. But since we know how to find the volume of the right prism, we obtain at once the following formula:

> The number of cubic units of volume in any prism is obtained from the formula
>
> $$V = Bh$$

where B is the number of square units of area in its base and h the number of linear units in its height.

7-5. Cylinders

Examples of cylinders are tin cans, water glasses, tanks, silos, and water pipes. The figures shown in the following drawing represent cylinders. Those on the left are called right cylinders. Compare them with the oblique (or slanted) cylinders on the right.



We rarely see slanted cylinders in ordinary life. Therefore in this chapter we shall assume our cylinders are right cylinders.

We list some important properties of a right cylinder.

(1) It has two congruent bases (a top and a bottom) and each is a circular region.

(2) Each base is in a plane and the planes of the two bases are parallel.

(3) If the planes of the bases are regarded as horizontal, then the upper base is directly above the lower base.

(4) The lateral or side surface of the cylinder is made up of the points of segments each joining a point of the lower circle with the point directly above it in the upper circle.

There are two numbers or lengths which describe a cylinder. These are the radius of the base of the cylinder and the height of the cylinder. The height is the (perpendicular) distance between the parallel planes containing the bases.

To find the volume of a cylinder whose radius has a measure 7 and whose height has a measure of 10:

$$V = Bh; \text{ for a cylinder, } B = \pi r^2$$

$$V = \pi r^2 h$$

Therefore, $$V \approx \frac{22}{7} \cdot 7 \cdot 7 \cdot 10$$

$$V \approx 1540$$

The volume of the cylinder is about 1540 cubic units.

In considering the surface area of a cylinder, two questions might be asked: (1) What is the total surface area? (2) What is the area of the curved surface? The total surface area is the area of the curved surface plus the area of the two bases. The areas of each of the bases are the same: πr^2, where r is the measure of the radius of the base. It remains for us to determine the area of the curved surface.

The lateral area of the cylinder is the area of a rectangular region. The length of the "rectangle" is actually the circumference of the cylinder, and the width of the "rectangle" is the height of the cylinder. The formula for the area of a rectangle is

$$A = l\,w,$$

for the cylinder, $$l = \pi d \text{ or } 2\pi r$$

and $$w = h$$

Therefore, the lateral area of a cylinder is the product of the measure of the length of the base circle and the measure of the height. This is commonly written as:

$$A = \pi dh \text{ or } A = 2\pi rh.$$

The measure of the total area may be found by using the formula commonly written:

$$S_T = \pi dh + 2\pi r^2 \text{ or } S_T = 2\pi rh + 2\pi r^2.$$

S_T represents the measure of the total surface of a cylinder,

d represents the measure of the diameter of the base,

r represents the measure of the radius of the base.

Exercises 7-5

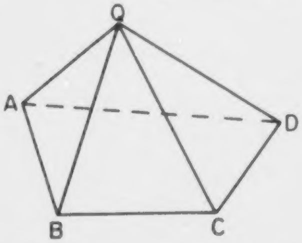
1. Information is given for three cylinders. Using $\pi \approx 3.1$, find all the missing data.

	Cylinder	Radius of base	Diameter of base	Height	Total Surface Area (At)	Volume (V)
(a)	C	?	16 ft.	17 ft.	?	?
(b)	D	15 cm.	?	50 cm.	?	?
(c)	E	?	8 yd.	12 yd.	?	?

2. A silo (with a flat top) is 30 feet high and the inside radius is 6 feet. How many cubic feet of grain will it hold? (What is its volume?) Use $\pi \approx 3.14$.
3. Find the volume of a cylinder whose height is 10 centimeters and the radius of whose base is 3 centimeters. Leave your answer in terms of π.
4. In Problem 3, what would the volume be if the height were doubled (and the base were left unchanged)?
5. In Problem 3, what would be the volume if the radius of the base were doubled (and the altitude were left unchanged)?
6. In Problem 3, what would be the volume if the height were doubled and the radius of the base were also doubled? (Think of first doubling the height and then doubling the radius of this new cylindrical solid.)
7. A small town had a large cylindrical water tank that needed painting. A gallon of paint covers about 400 square feet. How much paint is needed to cover the whole tank if the radius of the base is 8 feet and the height of the tank is 20 feet? Give your answer to the nearest tenth of a gallon.

7-6. Pyramids

Models 8 and 9 are for a figure obtained by joining the vertices of a polygon to a point not in the plane of the polygon, thus forming triangles. The pyramid consists of the closed triangular regions and the closed region of the original polygon. The closed region of the original polygon is called the base of the pyramid and the other faces its lateral faces. The point to which the vertices of the polygon are joined is called the apex of the pyramid. (Many books call this the vertex of the pyramid, but the term apex is chosen here since each corner of the polygon is also called a vertex.) The edges meeting at the apex are called lateral edges. For example, in the figure, the base is the interior of quadrilateral ABCD; the lateral faces are the closed regions of the triangles ABQ, BCQ, CDQ, and DAQ; the lateral edges are $\overline{AQ}$, $\overline{BQ}$, $\overline{CQ}$, $\overline{DQ}$; and the apex is Q.

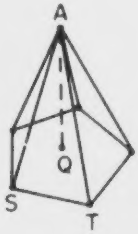


Imagine the segment drawn from the apex perpendicular to the plane of the base. This segment is called the altitude and the length of the altitude is the height of the pyramid.

Certain pyramids are called regular pyramids. To be regular, a pyramid must meet two conditions. First, its base must be the closed region of a regular polygon (having congruent sides and congruent angles), and second the foot of the altitude must be the center of this regular polygon. Then the lateral edges will all be congruent.

Exercises 7-6

1. Look at the figure which is supposed to show a regular pentagonal pyramid with apex A and altitude $\overline{AQ}$. Since Q is the center of the pentagon it is the same distance from S and from T. Suppose $\overline{AQ}$ is 4 inches long and $\overline{QT}$ and $\overline{QS}$ are each 3 inches long.



   (a) How can you find the lengths of $\overline{AS}$ and $\overline{AT}$?
   (b) What are these lengths?
   (c) Do $\overline{AS}$ and $\overline{AT}$ have equal lengths?
   (d) Is triangle AST isosceles?
   (e) Can the reasoning above be used to show that all five of the lateral edges have the same length?

2. Look again at the figure of Problem 1, with the base a regular pentagonal region, but this time suppose you know that the lateral edges all have the same lengths but do not know where the foot Q of the altitude is located. To be definite, suppose the height of the prism (i.e. length of $\overline{AQ}$) is 12 inches, and that each of the lateral edges $\overline{AS}$ and $\overline{AT}$ is 13 inches long.
   (a) How can you find the lengths of $\overline{QS}$ and $\overline{QT}$?
   (b) What are these lengths?
   (c) Are they equal?
   (d) Can this reasoning be used to show that the distances from Q to all five vertices of the polygon are equal?
   (e) Does this show Q is the center of the regular polygon?
   (f) Is the pyramid a regular pyramid?

7-7. Volumes of Pyramids

To find the volume of a pyramid, you must compare it with some figure whose volume you know. Construct with stiff paper or cardboard a regular square pyramid and a rectangular right prism whose bases are congruent and whose heights are equal. Fill the pyramid with salt and pour it into the rectangular prism. Do this several times until the rectangular prism is full. On the basis of this experiment do you agree with the following property?

Property 5. The volume of a pyramid whose base is congruent to the base of the prism and whose height is the same as that of the prism is one third that of the prism.

Since you know how to find the volume of a prism this leads at once to a formula for finding the volume of any pyramid:

$$V = \frac{1}{3} Bh$$

where B stands for the number of square units of area in the base and h the number of linear units in the height.

Exercises 7-7

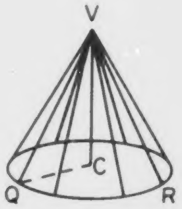
1. Find the volume of the pyramids, the measurements of whose bases and heights are as follows:
   (a) area of base = 12 square inches, height = 7 inches.
   (b) area of base = 100 sq. cm., height = 24 cm.

2. A pyramid has a square base of $1\frac{1}{2}$ inches on a side and a height of 4 inches. Find its volume.

3. What is the height of a pyramid whose volume is 324 cu. m. and whose base is a square, 9 m. on a side?
4. The Pyramid of Cheops in Egypt is 480 ft. high, and its square base is 720 ft. on a side. How many cu. ft. of stone were used to build it? (Assume that the pyramid is solid.) How many cubic yards?
5. Find the total area of the regular triangular pyramid whose lateral edge is 12 inches.
6. The side of the square base of a pyramid is doubled. The height of the pyramid is halved. How is the volume affected?

7-8. Cones

Anyone who has eaten an ice cream cone has at least a rough idea of the figure called a cone, or more strictly a right circular cone. Let a circle be drawn as shown below, with center C, and let V be a point not in the plane of the circle so that segment $\overline{VC}$ is perpendicular to this plane.



If all the segments from V were drawn to the points of the circle, the union of all these segments, together with the closed circular region, forms a right circular cone. The closed circular region is called the base of the cone, and the union of the segments is its lateral surface. The point V is called the vertex of the cone. In the description right circular cone the word circular indicates that the base is the closed circular region and the word right means that $\overline{VC}$ is perpendicular to the plane of the circle. Here only right circular cones will be considered and when the word "cone" is used it will mean this type.

Segment $\overline{VC}$ is called the altitude of the cone, and the length of this segment is the height of the cone. If Q is a point of the circle, what kind of triangle is VCQ? Why? If you know the height of the cone and the radius of its base can you find the length of $\overline{VQ}$? How? If R is another point of the circle, do $\overline{VQ}$ and $\overline{VR}$ have the same length? This constant distance from vertex V to the different points of the circle is called the slant height of the cone.

If h is the number of linear units in the height of the cone, r the number of linear units in the radius, and s the number of linear units in the slant height, write an equation relating h, r, and s. If you know any two of these numbers can you find the third one from this equation?

As an example, suppose the radius of the base of a cone is 10 in. and the height is 24 in. What is the slant height of the cone? Did you find the slant height to be 26 in.?

By experimentation one would be lead to accept the following property:

Property 6. The volume of the interior of a cone is one third that of a cylinder of the same height and whose base has the same radius.

Since you have already learned how to find the volume of a cylinder, this leads at once to the formula for finding the volume of a cone:

$$V = \frac{1}{3} \pi r^2 h.$$

Since πr^2 is actually the number of square units B of area in the base, the formula could be written as

$$V = \frac{1}{3} Bh$$

This gives the same rule for finding tne volume of a cone as for a pyramid.

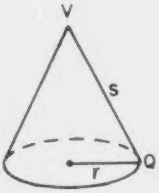
As an example, refer back to the cone mentioned above where the radius of the base was 10 inches and the height 24 inches. Then $r = 10$, $h = 24$, so by the formula above

$$V = \frac{1}{3} \pi (10)^2\ 24 = 800\ \pi\ ,$$

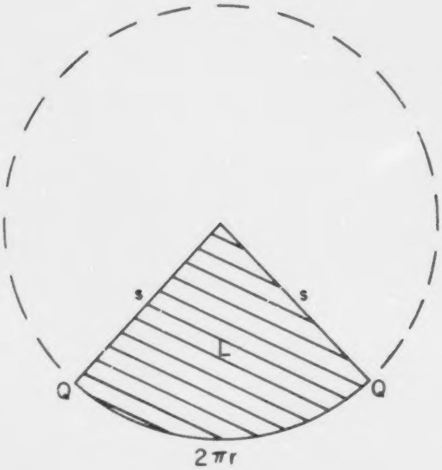
and the volume is $800\ \pi$ cu. in. or about 2512 cu. in.

Lateral Area of a Cone

To find the lateral area of a cone, look at Model 10. If we take it apart again, the lateral surface goes back into a sector of a circle as shown in the pattern for the model. (Notice that a sector of a circle is bounded by two radii and a part of the circle.) That is, the model which looks like this



flattens out into a sector of a circle that looks like this.



The lateral area of the cone has the same measure as the area of the shaded part we are trying to find. The two points marked Q in the figure come from the same point of the model. The rest of the large circle is shown in dotted lines to help you follow the reasoning.

Let s be the number of units in the slant height of the cone and r be the number of units in the radius of its base. Do the markings on the figure above on the two segments and the arc show the correct number of units in their lengths? Why?

Now in a sector of a circle, such as we have on the previous page, the area is related to the arc. For example, if the arc between the two points marked Q is one quarter of the circle, then the shaded area is one quarter of the interior of the circle. But the circumference of the circle is $2\pi s$ and its area is πs^2. If L stands for the number of square units in the shaded area, we find therefore, the proportion:

$$\frac{2\pi r}{2\pi s} = \frac{L}{\pi s^2}$$

If you multiply both sides of the equation of πs^2, what value do you find for L?

This reasoning justifies the following conclusion:

Property 7. If the slant height of a right circular cone is s units and the radius of its base r units, the number L of square units in its lateral area is given by the formula:

$$L = \pi rs.$$

As an example refer again to the cone where the radius of the base is 10 inches long and the height 24 inches. You recall we found the slant height to be 26 inches. In this problem we have, therefore, $r = 10$; $s = 26$, so

$$L = \pi\, 10 \cdot 26 = 260\,\pi \approx 816.4$$

and the lateral area is about 816.4 square inches.

Exercises 7-8

1. If T stands for the number of square units in the total area of the cone (counting the base) write a formula for T in terms of r and s.
2. The slant height of a cone is 12 ft. and the radius of its base 3 ft. Find its lateral area and its total area in terms of π.
3. A cone has a height of 12 ft. and its slant height is 15 ft. Find the radius, the lateral area, the total area, and the volume.
4. The radius of the base of a cone is 15 inches and the volume is 2700 cubic inches. Find its height, slant height, and lateral area.

Summary of Properties

Property 1. If a line is perpendicular to two distinct intersecting lines in a plane, it is perpendicular to the plane.

Property 2. The shortest segment from a point Q outside a plane r to the plane r is the segment perpendicular to that plane.

Property 3. If two planes are parallel the distances (perpendicular) from different points of one plane to the other plane are all the same.

Property 4. If two prisms have congruent bases and equal heights they have equal volumes.

Property 5. The volume of a pyramid is one-third that of a prism whose base is congruent to the base of the prism and whose height is the same as that of the prism. ($V = \frac{1}{3}Bh$, where B stands for the number of square units of area in the base and h the number of linear units in the height.)

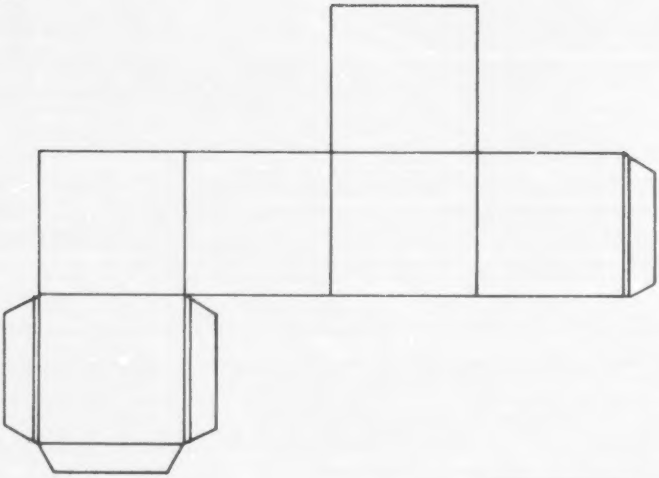
Property 6. The volume (of the interior) of a cone is one third that of a cylinder of the same height and whose base has the same radius. ($V = \frac{1}{3}\pi r^2 h$ or $V = \frac{1}{3}Bh$)

Property 7. If the slant height of a right circular cone is s units and the radius of its base r units, the number L of square units in its lateral area is given by the formula,

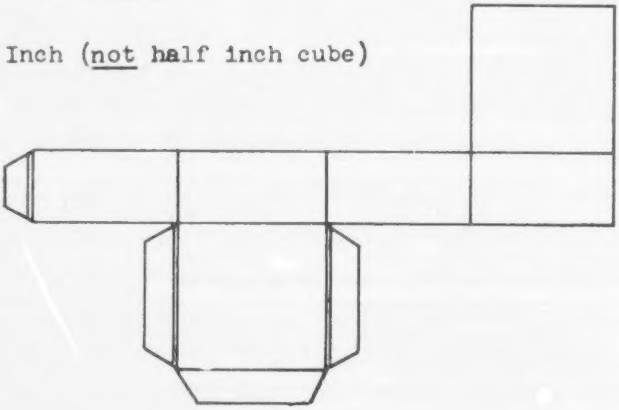
$$L = \pi rs.$$

Notice that in the models the exact measurements are not the measurements indicated - they are merely drawn to scale.

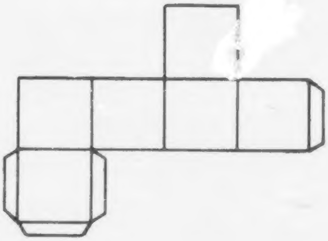
Model 1 - Inch - Cube



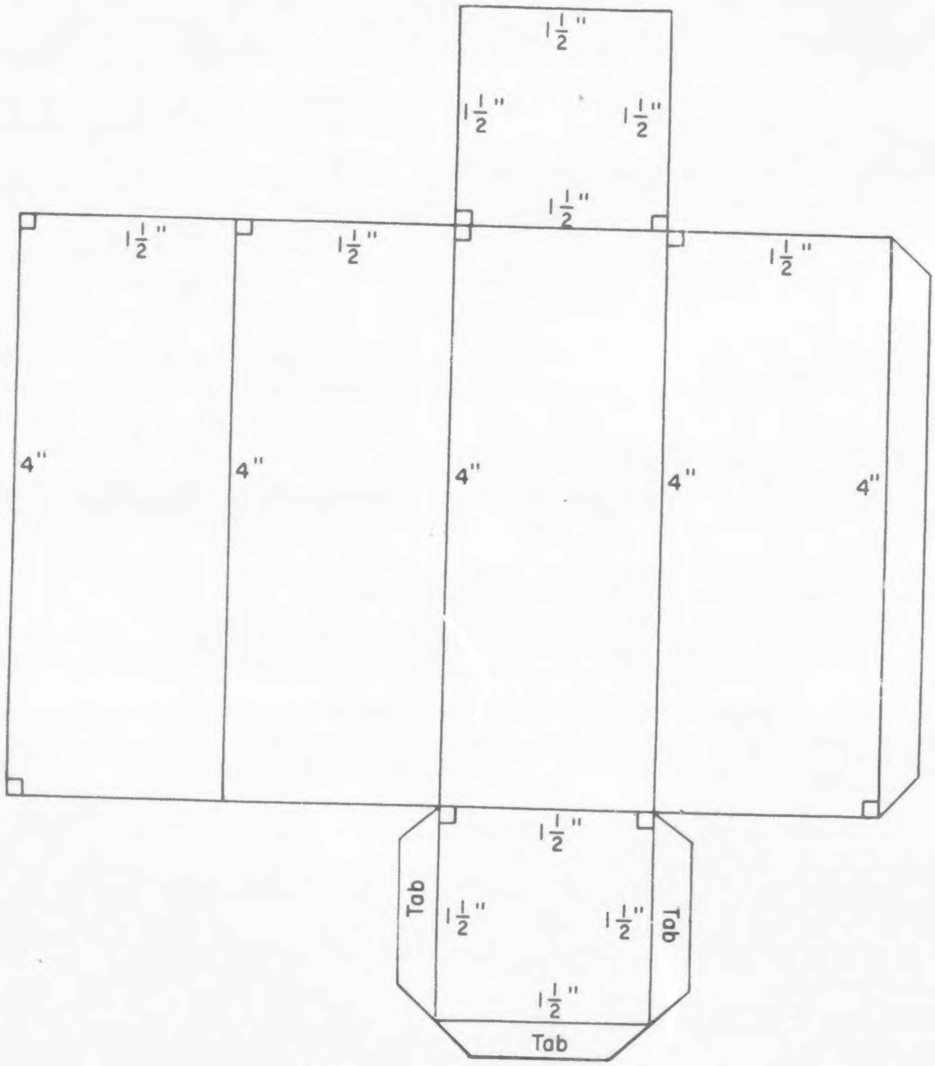
Model 2 - Half Cubic Inch (not half inch cube)



Model 3 - Half-inch cube.



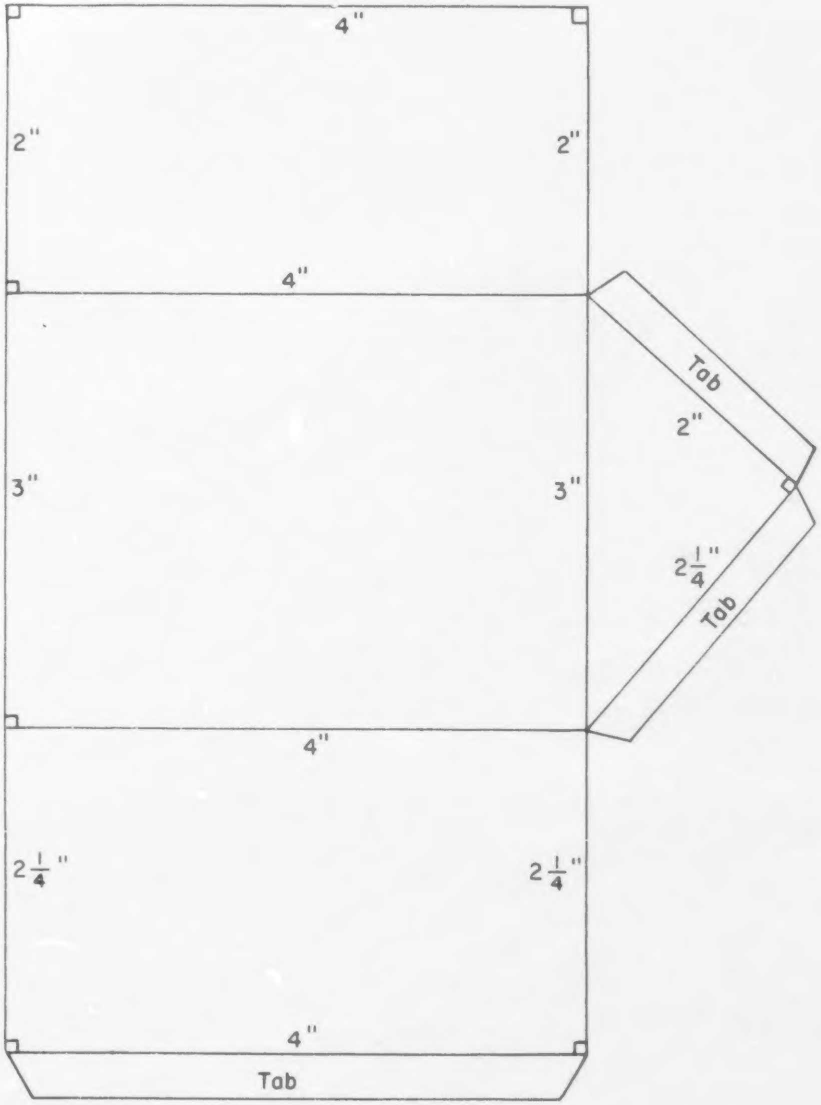
Model 4 - Rectangular Right Prism



Model 5 - Right Triangular Prism.

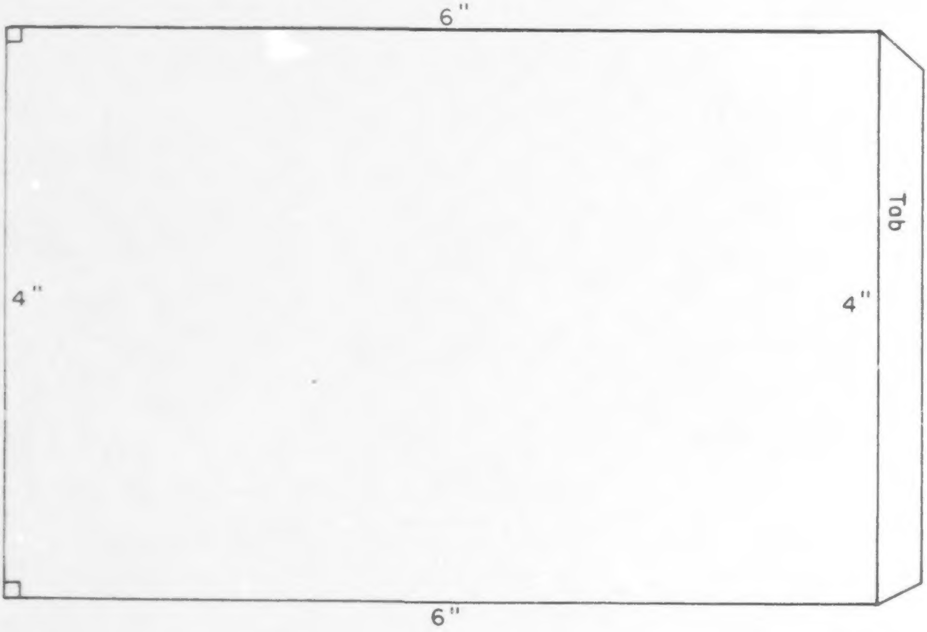
(Base is Interior of a Right Triangle)

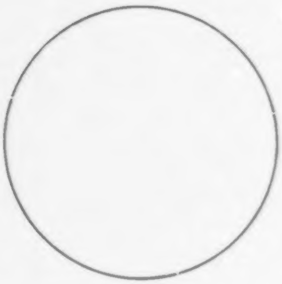
Make an extra copy of the base to use for the other base, but use only one tab so the top can be opened.



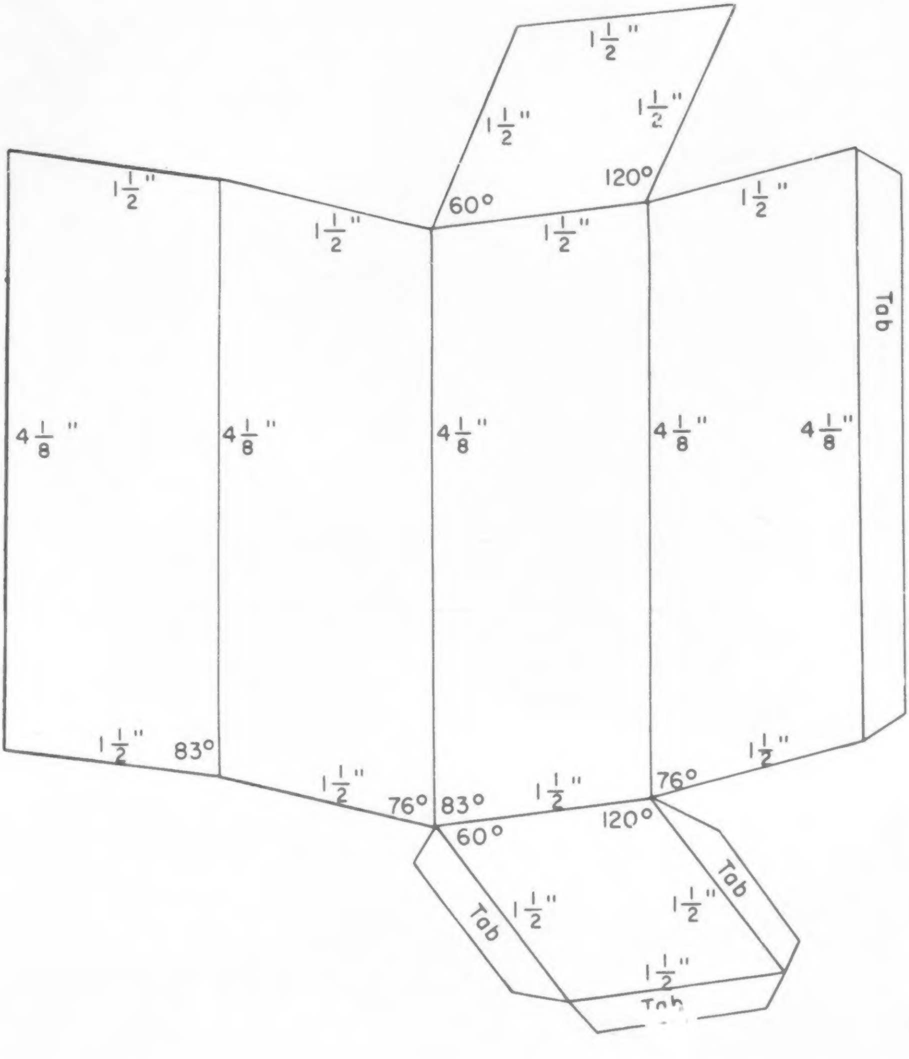
Model 6 - Right Circular Cylinder

It will be easier to draw the circular bases with your own compass using the radius of the circle below rather than trying to trace the circle as shown. Make two copies of the circle, since there are two bases. Attach the lower base firmly (with tape) but attach the top base only at one point so it can be readily opened.

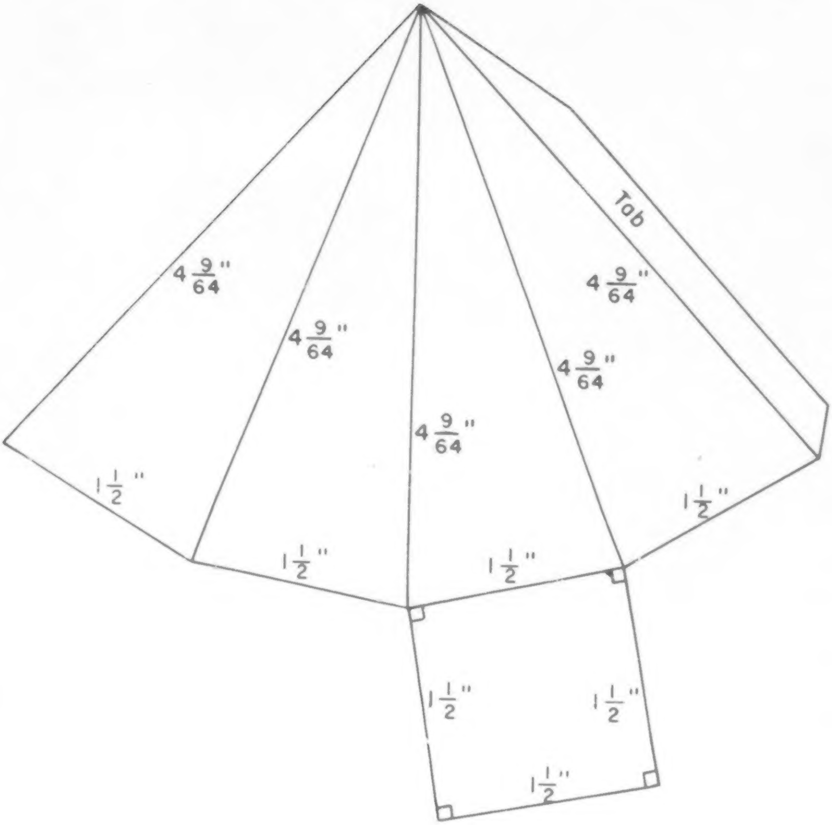




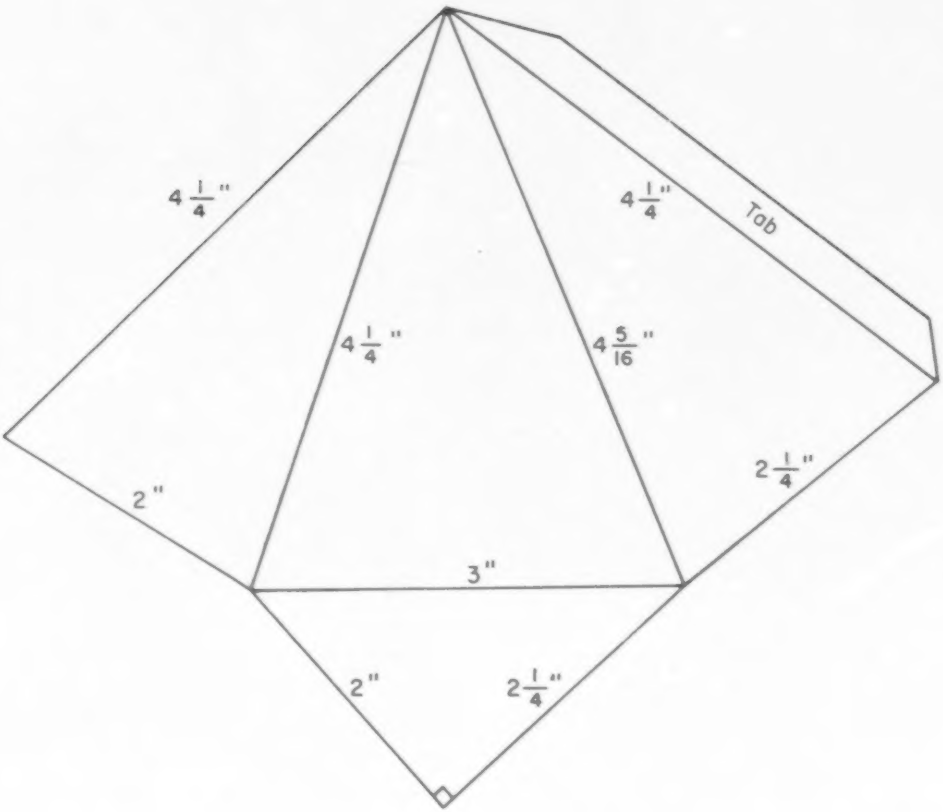
Model 7 - Oblique Prism with Rhombus as Base
(Also Parallelepiped)



Model 8 - Regular Square Pyramid

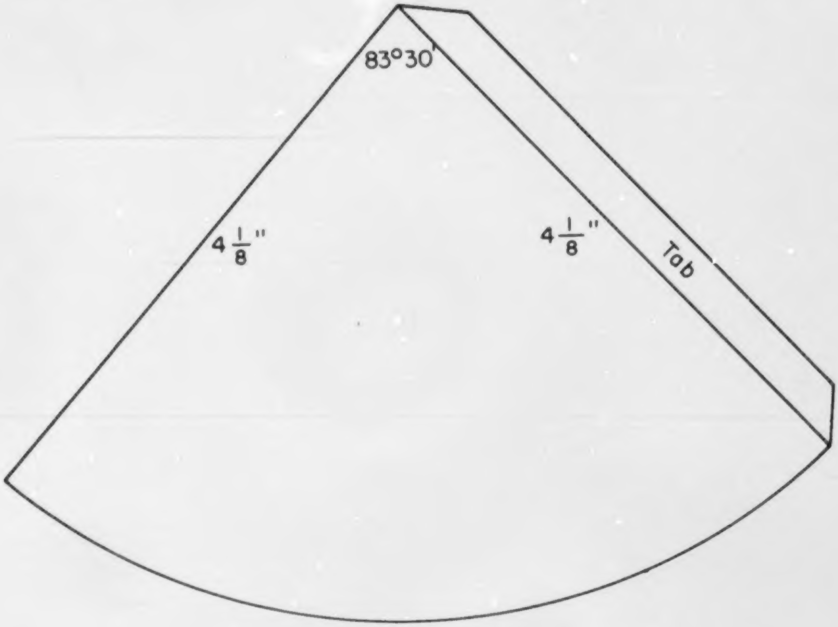


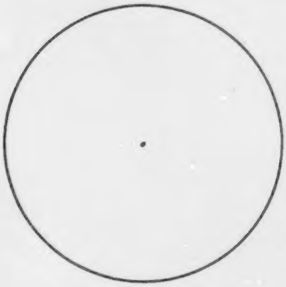
Model 9 - Triangular Pyramid (Tetrahedron)



Model 10 - Right Circular Cone

It will be better to draw these circles with your own compass using the radii of the circles drawn rather than trying to trace them. The radius of the small circle is supposed to be the same as in Model 6.





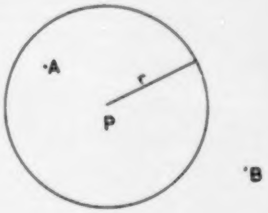
Chapter 8

CIRCLES AND SPHERES

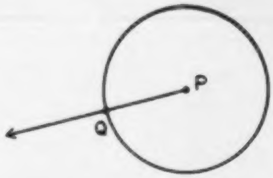
8-1. Interiors and Intersections

The circle with center P and radius r units is the set of all points in a plane at a distance r from P.

A circle is a simple closed curve and thus has an interior and an exterior. In the drawing at the right, the interior is the set of all points at distance less than r units from P. The exterior is the set of all points at distance greater than r units from P. Point A is in the interior and Point B is in the exterior of circle P.

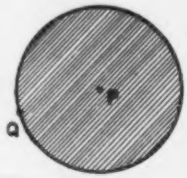


The drawing at the right shows a ray $\overrightarrow{PQ}$ with end-point on P, the center of the circle. How many points of the circle are on the ray $\overrightarrow{PQ}$? How many points of the circle are on the ray $\overrightarrow{QP}$?



How would you describe the location of the additional point determined by $\overrightarrow{QP}$ which is not on $\overrightarrow{PQ}$?

The interior of a simple closed curve is called a "closed region." In the drawing at the right the interior of the circle is the shaded region. The union of a circle and its interior is called a circular closed region.



The union of a circle and its interior is the set of all points whose distance from the center P is either the same as or less than the radius of the circle. What is the intersection of a circle and its interior? No point of the circle also lies in the interior of the circle. This intersection is the empty set.

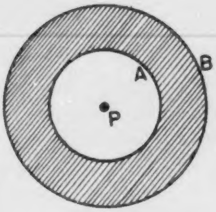
Let Y represent the union of circle P and its interior. Q is a point on the circle. Note that $Y \cap \overleftrightarrow{PQ}$ is quite different from the intersection of circle P and $\overleftrightarrow{PQ}$.

Exercises 8-1

In the following exercises, draw the figures:

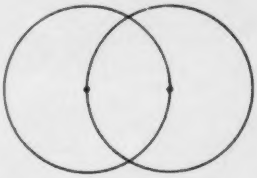
1. Let C be a circle with center at P and radius r units. Let S be any other point in the same plane.
   (a) How many points belong to the intersection of the circle C and the ray $\overrightarrow{PS}$?
   (b) How many points belong to the set $C \cap \overleftrightarrow{PS}$?
   (c) Do your answers to (a) and (b) depend upon where you chose the point S?
   (d) How many points belong to the set $C \cap \overline{PS}$? Does this answer depend on the choice of S? If so, how?
2. In a plane, can there be two circles whose intersection consists of just one point?
3. Choose two points and label them P and Q. Draw two circles with center at P such that Q is in the exterior of one circle and in the interior of the other. Label the first circle C and the second circle D.
4. Choose two distinct points P and Q. Draw the circle with center at P and with the segment $\overline{PQ}$ as a radius. Then draw the circle with center at Q and with P on the circle.
   (a) What is the intersection of these two circles?
   (b) Can you draw a line which passes through every point of the intersection of the two circles? Can you draw more than one such line? Why?
   (c) In your picture shade the intersection of the interiors of the two circles. (If you have a colored pencil handy, use it for shading; if you do not, use your ordinary pencil and shade lightly.)
   (d) (In this part, use a different type of shading or, if you have one handy, use a pencil of a different color.) Shade the intersection of the interior of the circle whose center is P and the exterior of the circle whose center is Q. (Before doing the shading of the intersection, you may find it helpful to mark separately the two sets whose intersection is desired.)
   (e) Make another copy of the picture showing the two circles, and on it shade the union of the interiors of the two circles.

5. The two circles shown at the right lie in the same plane and have the same center, P. Circles having the same center are called concentric circles.



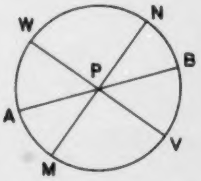
   (a) Describe the intersection of circle A and circle B.
   (b) Give a word description of the shaded region, using such words as "intersection," "exterior," and so on.

6. In the drawing at the right, the center of each circle lies on the other circle. Copy the figure on your paper. Shade the union of the exteriors of the two circles.



8-2. Diameters and Tangents

A diameter of a circle is a line segment which contains the center of the circle and whose end-points lie on the circle. In the drawing shown at the right, three diameters are shown: $\overline{AB}$, $\overline{MN}$, and $\overline{VW}$. A diameter is the longest line segment which is contained in a circular closed region. The length of any diameter of a circle is also referred to as the diameter of the circle.

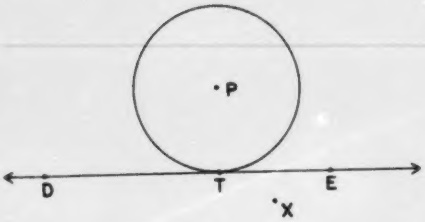


A diameter may also be described as the union of two different radii which are segments of the same line. Using any unit of length, if we let r and d be the measures of the radius and the diameter of a circle respectively, we have these important relationships:

$$d = 2r \quad \text{and} \quad r = \frac{1}{2}d.$$

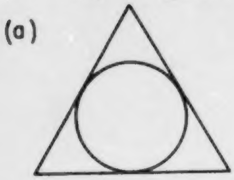
A tangent to a circle is a line that intersects the circle in exactly one point. In the drawing $\overleftrightarrow{DE}$ is tangent to circle P.

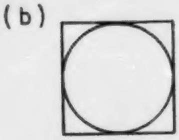
The single point of their intersection is T. It is called the point of tangency. A tangent to a circle cannot contain a point of the interior of the circle.

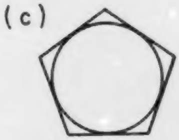


Exercises 8-2

1. How many tangents do you find in each of the following?







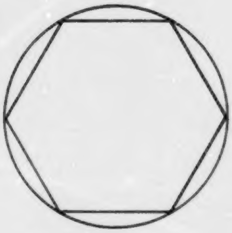
2. Find the diameters for each of the circles where the distance from the center of the circle to a point on the circle is as follows:

   (a) 6 in. (c) 17 cm. (e) $3\frac{1}{2}$ ft.

   (b) 3 m. (d) 5 ft.

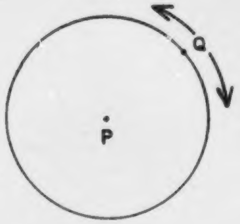
3. Draw a circle C with center at the point P. Draw three diameters of C. Draw a circle with center at P whose radius is equal to the diameter of C.

4. With compass and straight edge copy the drawing shown at the right. Note that the sides of the hexagon and the radius are congruent.

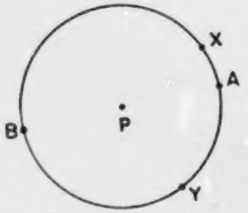


8-3. Arcs

A point on a line separates the line into two half lines. This is not true for simple closed curves. In the drawing at the right, starting at Q and moving in a clockwise direction along the circle, we return to Q. The same is true if we move in a counter clockwise direction.



In the figure at the right, the two points, X and Y, separate the circle into two parts. One of the parts contains the point A. The other part contains B. No path from X to Y along the circle can avoid at least one of the points, A and B. Thus, we see it takes two different points to separate a circle into two distinct parts.

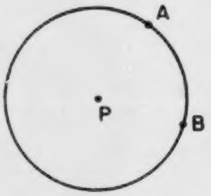


Note in the above drawing that while Point A is between X and Y, point B is also between X and Y. For circles, or other simple closed curves, we observe:

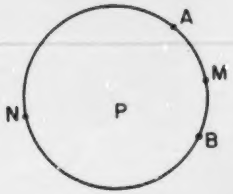
(1) A single point does not separate the curve into two parts.

(2) Separation and betweenness are not closely related notions for simple closed curves.

A part of a circle is called an arc. In the drawing at the right, points A and B separate the circle into two arcs. Each of these arcs have points A and B as end points. The arc includes its endpoints. The arc, starting at A and moving clockwise to B is shorter than the arc starting at A and moving counter-clockwise to B.



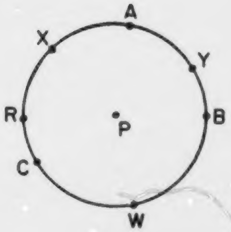
The symbol "⌒" represents the word "arc." With only two points on a circle it is difficult to identify one of the two arcs determined by A and B. Usually a third point is marked and labeled between the two points. $\overset{\frown}{AMB}$ (or $\overset{\frown}{BMA}$) represents the arc containing point M.



Exercises 8-3

1. In the drawing on the right, identify the shortest arc containing the following points, where the points are not endpoints of the arc.

   (a) A
   (b) B
   (c) C
   (d) R
   (e) X
   (f) Y



2. What are the endpoints for these arcs from the drawing in Problem 1?

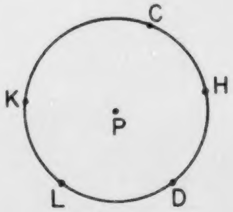
   (a) $\overset{\frown}{AYB}$
   (b) $\overset{\frown}{AXR}$
   (c) $\overset{\frown}{ACB}$
   (d) $\overset{\frown}{YWC}$
   (e) $\overset{\frown}{WRA}$
   (f) $\overset{\frown}{BAC}$

3. In answering the previous question, was it necessary to see the drawing? Explain.

4. Use the drawing at the right in answering the following:

   (a) Point L separates $\overset{\frown}{CLH}$ into two arcs. Name these two arcs.

   (b) Does point L separate the circle into two arcs? Explain why, or why not.



5. Use your answers in Problem 4, above, to answer the following:

   (a) Does a point on an arc separate the arc into two arcs?

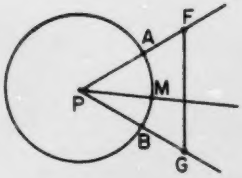
   (b) On an arc, must a point, which is not an endpoint of the arc, be between two points of the arc?

   (c) Does an arc have a "starting" point and a "stopping" point? If so, what are they called?

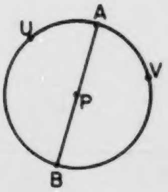
   (d) For an arc, are the notions of betweenness and separation more like those of a line segment or of a circle?

8-4. Central Angles

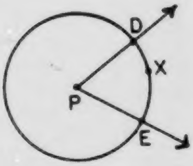
Arcs have some properties similar to properties of line segments. At the right note the natural one-to-one correspondence between the set of points of $\overset{\frown}{AMB}$ and the set of points of $\overline{FG}$. On a line segment, a point, not an endpoint, separates the segment into two parts. Similarly, an arc is separated into two arcs by a point which is not an endpoint of the arc. As with a segment an arc has endpoints.



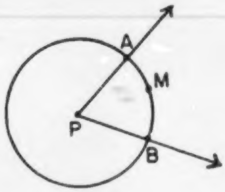
In the drawing at the right, the endpoints of a diameter are A and B. These points determine two special arcs called semi-circles. In the drawing, $\overset{\frown}{AVB}$ is a semi-circle.



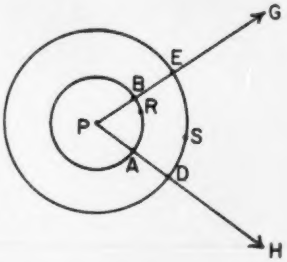
The endpoints of a semi-circle and the center of the circle are on a straight line. For other arcs this is not true as shown at the right. The center P is not on the straight line passing through the endpoints of $\overset{\frown}{DXE}$. Rays $\overrightarrow{PD}$ and $\overrightarrow{PE}$ determine a central angle. A central angle is an angle having its vertex at the center of a circle. Such angles are measured in the same way as other angles.



To measure arcs we use a unit called one degree of arc. We think of a circle divided into 360 congruent arcs. Each such arc determines a unit of arc measure. Rays from the center of a circle passing through the endpoints of such an arc determine a central angle. One degree of arc is determined by a central angle which is



a unit angle of one degree. In the figure at the right, if the measure of $\widehat{AMB}$ in degrees of arc is 80, then the measure of $\angle APB$ in angle degrees is 80. The symbol for a degree of arc, "o", is the same as that for the angle degree.

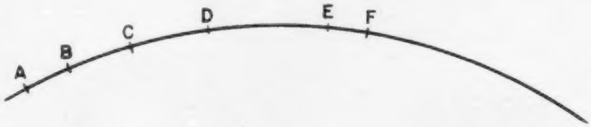


In the figure at the right are two circles having a common center, P. The circles are in the same plane. Such circles we have called concentric circles. The two arcs, $\widehat{ARB}$ and $\widehat{ESD}$ have the same central angle, $\angle GPH$. Therefore, $\widehat{ARB}$ and $\widehat{ESD}$ must have the same arc measure. If the angle measure of $\angle BPA$ is 70, then the arc measure of $\widehat{ARB}$ is 70. The arc measure of $\widehat{ESD}$ must also be 70. However, $\widehat{ARB}$ appears shorter than $\widehat{ESD}$. Remember that arc measure is not a measure of length. Two arcs may have the same arc degree measure but have different **lengths.** The reason will be more apparent after you have studied the remainder of this chapter.

Exercises 8-4

1. Construct a circle with radius approximately $1\frac{1}{2}$ inches. In this exercise mark off the points in a counter-clockwise path around the circle after starting anywhere on the circle with A. Mark off and label arcs with the following measures:
    - (a) $m(\widehat{AB}) = 10$
    - (b) $m(\widehat{AC}) = 45$
    - (c) $m(\widehat{BD}) = 50$
    - (d) $m(\widehat{DF}) = 170$
    - (e) What is $m(\widehat{BC})$?
2. (a) How many arc degrees are in a quarter of a circle?
    - (b) How many arc degrees are in one-eighth of a circle?
    - (c) How many arc degrees are in one-sixth of a circle?
    - (d) How many arc degrees are in three-fourths of a circle?

3. Refer to the arc $\widehat{ABCDEF}$, or more briefly $\widehat{AF}$, shown in the drawing below. Determine the following:



(a) $\widehat{AC} \cap \widehat{BD}$

(b) $\widehat{AF} \cap \widehat{DF}$

(c) $\widehat{AD} \cap \widehat{CF}$

(d) $\widehat{CD} \cap \widehat{DE}$

(e) $\widehat{DF} \cap \widehat{AE}$

4. Circle A has a radius of 5 inches. Circle B has a radius of 25 inches. Explain why the arc measure of one-fourth of circle A is the same as the arc measure of one-fourth of circle B.

5. Demonstrate a one-to-one correspondence between the sets of points on the two semicircles of a given circle which are determined by a diameter.

8-5. Circumference of Circles

It is difficult to measure the circumference of a circle accurately. The circumference of a circle is the length of the simple closed curve which we call a circle. Experimentation with circular objects will reveal that the circumference is a little more than three times the diameter of the circle.

Mathematicians have proved that, for any circle, the ratio of the measure of the circumference to the measure of the diameter is always the same number. The special symbol "π" represents this ratio. The symbol is read "pi" and is the first letter in the Greek word for "perimeter." In mathematical language this relation is written as follows: $\frac{c}{d} = \pi$ or $c = \pi d$.

Exercises 8-5

1. Find the missing information about the circles described. (Use $\pi \approx 3.14$).

	Circle	Radius	Diameter	Circumference
(a)	C	3.6 yd.	?	?
(b)	D	4.2 cm.	?	?
(c)	E	?	5.6 in.	?
(d)	V	?	?	22
(e)	Z	?	?	88

2. Sometimes it is a good idea to use π as a numeral, instead of using a decimal for π. Answer the following question using π as a numeral. We say the answer is expressed "in terms of π ".

   If the length of a circle is 54π in., what is its diameter? Its radius?

3. A circle with a diameter of 20 inches is separated by points into 8 arcs of equal length.
   (a) What is the length of the whole circle?
   (b) What is the length of each arc?
   (c) What is the arc measure of each arc?
   (d) On this circle, how long is an arc of one degree?

The Number π

The number represented by the symbol "π" is a new kind of number. It is not a whole number. Neither is it a rational number. Recall that any decimal expansion of a rational number is a repeating decimal. Mathematicians have proved that π cannot be a repeating decimal. In an article by F. Genuys in Chiffres I (1958), a decimal expression for π to 10,000 decimal places was published.

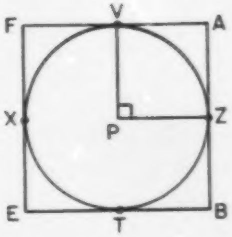
Here is the decimal for π to fifty-five places.

3.14159 26534 89793 23846 26433 83279 50288 41971 69399 36510 58209...

(The three dots at the end indicate that the decimal expression continues indefinitely.)

8-6. Area of a Circle

When speaking about a circle in everyday language, we usually use the phrase "the area of a circle" when we mean "the area of the closed circular region."



In the drawing, the point P is the center of the circle and also the center of the square ABEF. Let the measure of a radius of the circle be r. Then the segments $\overline{VP}$ and $\overline{PZ}$ are radii with measure r. Angle VPZ is a right angle. The area of the square is four times as large as the area of square VAZP. Note that the area of the circle is less than that of square ABEF. Similarly, it is less than the area of four squares such as VAZP. The area of the circle is a little more than three times as great as the area of VAZP. Compare the area of the circle with the area of the square ABEF.

Mathematicians have proved that for any circle, the area of the circle is a little more than three times the measure of the radius multiplied by itself. In mathematical language,

$$A = \pi r^2$$

where A is the number of units of area and r is the measure of the radius. The area of a nine-inch circle (having a radius of $4\frac{1}{2}$ in. or $\frac{9}{2}$ in.) may be found as follows:

$$A = \pi r^2 \quad (\text{Using } \pi \approx 3\tfrac{1}{7})$$

$$A \approx \frac{22}{7} \cdot \frac{9}{2} \cdot \frac{9}{2}.$$

$$A \approx \frac{1782}{28} \text{ or } 63\tfrac{18}{28}.$$

Exercises 8-6

1. Find the area of each circle for which the radius is given. (Use $\pi \approx 3.14$)

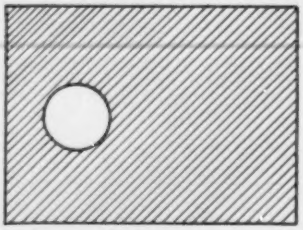
   (a) 8 ft. (b) 10 yd. (c) 15 cm.

2. Find all the missing data. Use 3.1 as an approximation to π.

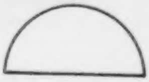
	Circle	Radius	Diameter	Circumference	Area
(a)	A	4 ft.			
(b)	C		20 ft.		
(c)	D			100 mi.	

3. Which has the greater frying surface--an eight-inch circular skillet or a seven-inch square frying-pan? ($\pi \approx 3.14$)

4. A rectangular plot of land, 40 feet by 30 feet, is mostly lawn. The circular flower-bed has a radius of 7 feet. What is the area of the portion of the plot that is planted in grass?



5. The figure represents a simple closed curve composed of an arc of a circle and a diameter of the circle. The area of the interior of this simple closed curve, measured in square inches, is 8π. Do not use any approximation for π in this problem.



   (a) What is the area of the interior of the entire circle?
   (b) What is the second power of the radius of the circle?
   (c) How long is a radius of the circle?
   (d) How long is the straight portion of the closed curve represented in the figure?
   (e) What is the circumference of the (entire) circle?
   (f) How long is the circular arc represented in the figure?
   (g) What is the total length of the simple closed curve?

6. The earth is about 150 million kilometers from the sun. The orbit (or path) of the earth around the sun is not really circular, but approximately so. Suppose that the orbit were a circle; then the path would lie in a plane and there would be an interior of the orbit (in the plane). What would be an estimate for the area of this interior?

8-7. The Sphere

What is suggested by "a set of points in space whose distances from a particular point are all the same?" This set of points would be more than a circle. It would be a surface, like the surface of a ball. Such a surface is called a sphere. The point from which the distances are measured is called the center of the sphere.

Many objects are spherical, i.e., have the shape of a sphere; rubber balls, used as toys; ball bearings, important to industry; the earth, which is far from being a perfect sphere.

Note that it is the surface that is called a sphere. On a ball, only that portion that could be painted represents the surface.

Let us consider the set of all lines which intersect a sphere in two distinct points (pass through a sphere.) Each such line contains a line segment whose end-points lie on the sphere. Are all such segments congruent? No, but one particular subset of this set consists of congruent segments, the set of line segments passing through the center of the sphere. A line segment whose end-points lie on the sphere and which passes through the center is called a diameter.

Exercises 8-7

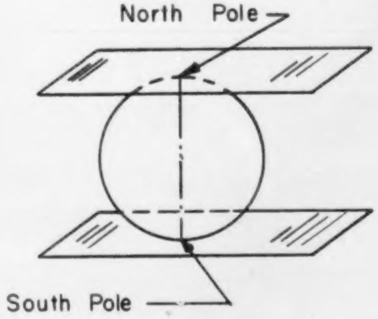
1. (a) Suppose all the points of a sphere are a distance v from the center, C, of the sphere. How can you describe the set of points such that all points are located a distance less than v from C?
   (b) How can you describe the set of points such that all points are located a distance greater than v from C?
2. In section 8-1 of this chapter, the interior of a circle was defined to be all points of a set, including the center itself, whose distance from the center is less than the radius.
   (a) Using the above definition as an example, define the interior of a sphere.
   (b) Similarly, define the exterior of a sphere.
   (c) Similarly, define a sphere.

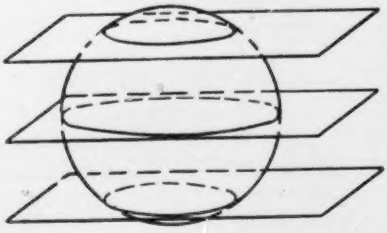
8-8. Great and Small Circles of a Sphere

The line passing through the poles of the earth is called the axis of the earth. This is approximately the line about which the earth revolves. Accepting the surface of the earth as a sphere, the diameter contained in the axis intersects the sphere at the North and South Poles. The points represented by these poles are "directly opposite" each other. Each diameter of the earth will contain two such points called *antipodal points*. The North Pole represents a point which is the antipode of a point represented by the South Pole.

We can connect any point, P, of a sphere with the center of the sphere. A line passing through these two points will intersect the sphere in a third point, the antipode of point P.

In the drawing at the right, assume that the horizontal planes just touch the sphere at the ends of the vertical axis shown. Such planes, intersecting a sphere at one point are said to be *tangent* to the sphere. An infinite number of such planes exist.





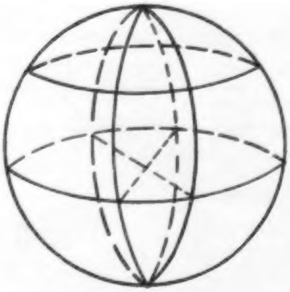
If we lower the top horizontal plane as shown in the drawing at the right, the intersection of the plane and the sphere will be a circle. These circles will vary in length. One such circle, determined by the plane passing through the center of the sphere, has a circumference greater than the others. On the earth we call this circle the *equator*. Many such circles may be determined, however, by other planes intersecting the sphere and passing through the center of the sphere. Such circles are called *great circles*, and the other circles are called *small circles*. We have, therefore, this definition:

Definition: A great circle on a sphere is any intersection of the sphere with a plane through the center of the sphere.

Definition: All circles on a sphere which are not great circles are called small circles.

All great circles on a sphere have the same length since their radii are equal to the radius of the sphere. The length of every great circle on a sphere is greater than the length of any small circle on that sphere.

Again, think of the earth as a sphere. We can imagine many great circles of this sphere. A particular set of great circles of the earth is the set whose members pass through the North and South Poles. Consider half of such a great circle. On the earth, half of a great circle, with the poles as end-points, is called a meridian. We sometimes use the term "semicircle" in talking about half of a circle.



The small circles whose planes are parallel to the plane of the equator are called parallels of latitude. These circles have their centers on the axis of the earth and their planes perpendicular to the axis of the earth.

Meridians and parallels of latitude will be discussed more carefully later. At that time we shall discuss how points on the surface of the earth can be located by means of these great and small circles.

In a plane, the shortest distance between two points is along a straight line. On a sphere this is not true, although it may appear to be true when you think of two points rather close together on the earth. A plane, flying from New York to San Francisco follows the curvature of the earth. On a sphere, it turns out to be true that the shortest distance between any two points is a path along a great circle that passes through the two points. (You may have heard of "the great circle route" for airplanes and ships.) However, by using a string stretched around a globe you may test this statement.

Exercises 8-8

You may find it helpful to use a globe representing the earth or a large ball on which drawings can be made in answering the following questions.

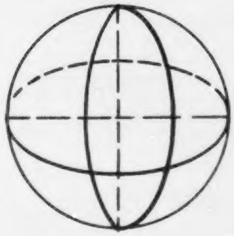
1. (a) Is there an antipode of any given point on a sphere?
   (b) Is there more than one antipode of any given point on a sphere?
2. (a) How many great circles pass through a given point, such as the North Pole, of a sphere?
   (b) How many small circles pass through a given point of a sphere?
   (c) Can a small circle pass through a pair of antipodal points on a sphere? Explain.
3. (a) On a sphere, does every small circle intersect every other small circle? Explain.
   (b) On a sphere, does every great circle intersect every other great circle? Explain.
4. (a) In how many points does each meridian cut the equator? Explain.
   (b) In how many points does each meridian cut each parallel of latitude?
   (c) Does a parallel of latitude intersect any other parallel of latitude? Explain.

8-9. Properties of Great Circles

For short trips, i.e. keeping within the boundaries of one of the states of the United States, an ordinary road map may be used as a guide. For trips between cities it is relatively easy to select a shortest route. For ocean travel, however, a road map is of little use. A globe would be more helpful.

To understand travel on the globe better, let us review some fundamental properties of spheres.

Property 1. Every pair of distinct great circles intersect in two antipodal points.



This property is easily proved as follows:

(1) Every great circle of a sphere lies on a plane through the center of the sphere.

(2) All planes containing a great circle have the center in common, and thus any two such planes must intersect.

(3) The intersection set of any two intersecting planes is a line.

(4) This line intersects the sphere in two antipodal points since the line passes through the center.

(5) Thus, these antipodal points are the points of intersection of the two great circles on the two planes.

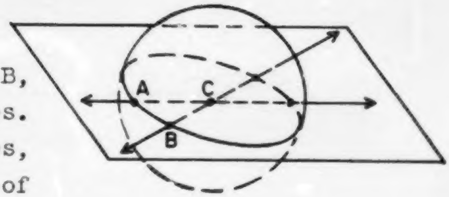
We will use this property in discussing distances between points on a sphere.

In the previous section we stated that the shortest distance on the surface of a sphere between any two points on the sphere is measured along the path of a great circle. In the study of geometry in high school mathematics this statement is proved. We will not do so now.

Suppose you are to travel from the North Pole to the South Pole. You should be able to find many possible "shortest" routes. Each meridian is a possible route. If we think of the earth as a sphere, the meridians are congruent. Thus, it does not matter which meridian is selected as your route. For any two antipodes, then, all great circle paths for the sphere being congruent, there are any number of paths one can take.

But, what if the two points are not antipodes? How many possible paths along a great circle route are there? There are only two possible great circle paths between two such points and both lie on the same great circle. This is the next important property.

Property 2. Through any two points of a sphere, which are not antipodes, there is exactly one great circle.



We can prove this property as follows:

(1) Think of any two points, A and B, on a sphere which are not antipodes.

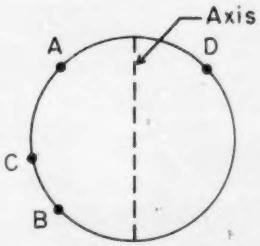
(2) Since A and B are not antipodes, a line through A and the center of the sphere C. cannot pass through B. (Similarly, $\overleftrightarrow{BC}$ does not contain A).

(3) Through the three points, A, B, and C there can pass exactly one plane, because these three points are not contained in one line.

(4) The one plane containing A, B, and C, contains only one great circle with center at C.

(5) This great circle is thus the only great circle passing through A and B.

This property tells us that if two points on the sphere are not antipodes, there is exactly one shortest route between these two points. Of course, there are two directions one can travel along a great circle containing A and B. In the drawing at the right we see that one route, ADB, would pass through D, the other, ACB, through C. Since A and B are not antipodes, one route must be shorter than the other. We naturally choose the one which is shorter. Can you pick the shortest route in the drawing?



From the point of view of shortest distance, the great circles on a sphere behave like straight lines on a plane. We have shown also that through any two points there is just one great circle unless the points are antipodal. But great circles on a sphere do not behave like straight lines in all respects for any two great circles intersect in two points. There are no parallel great circles on a sphere.

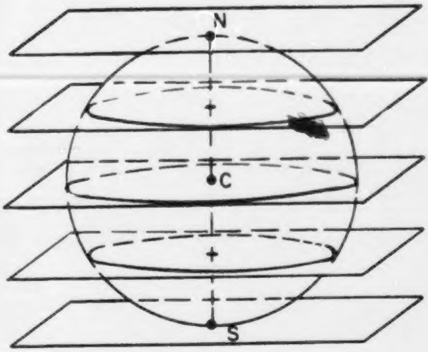
Exercises 8-9

Use a globe and length of string and a ruler to answer questions 1-3.

1. Locate Nome, Alaska and Stockholm, Sweden on the globe.
    - (a) Place one end-point of the string on the location of Nome. Place the string on a northern path, passing through the North Pole. Continue until you reach Stockholm. Carefully mark on the string a point which falls on the location of Stockholm. What is the distance on the globe in inches from Nome to Stockholm as represented by the segment marked on the string?
    - (b) Using a string and ruler, what is the distance from Nome to Stockholm along a route directly east from Nome?
    - (c) From your results above, what is the shorter distance between the two points represented by Nome and Stockholm, a path following a great circle; or a path following a small circle on the line of latitude?
2. (a) What is the distance from Nome, Alaska to Rome, Italy along a great circle route which passes through a point near the North Pole?
    - (b) What is the distance from Nome to Rome along a southeasterly course passing through the southern tip of Hudson Bay, and through a point on the border between Spain and France?
    - (c) How do your results in (a) and (b) compare?
3. A merchant living in Singapore, Malaya, plans to take a non-stop flight to Quito, Ecuador. What is the best route between these two points?
4. Explain why going due north would be the shortest although not necessarily the safest or best, route in traveling to a point on the earth directly north of your starting point.
5. (a) Explain why going due east is not always the most efficient way of getting to a point directly to the east.
    - (b) When is a route due east or west always the most efficient?
6. Given three points on a sphere. Can a circle on the sphere (small or great) be drawn through all three points?

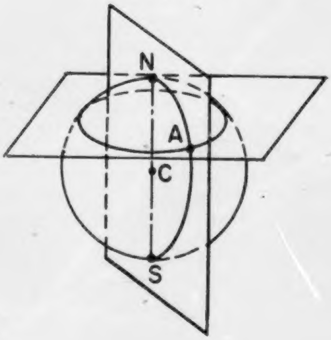
7. BRAINBUSTER: A hunter set out walking due south from his camp. He walked for about two hours without seeing any game. Then he walked 12 miles due east. At this point he saw a bear which he shot. To return to camp he traveled directly north. What color was the bear? (Note: this problem does have an answer.)

8-10. Locating Points on the Surface of the Earth.



These are parallel planes slicing the earth in horizontal sections as shown at the right. The top plane is tangent to the North Pole, and the bottom plane is tangent to the South Pole. The intersection of each of the remaining planes and the earth is a circle. The circles determined by these planes are all small circles except for the equator, which is a great circle. All such circles are called parallels of latitude. They are called "parallels" because they are determined by planes parallel to the plane passing through the equator.

The second set of curves consists of the meridians. Recall that meridians are halves of great circles such that the half-circles have the poles as end-points. Thus, each great circle through the poles consists of two meridians. Each meridian has a diameter, the axis of the earth.

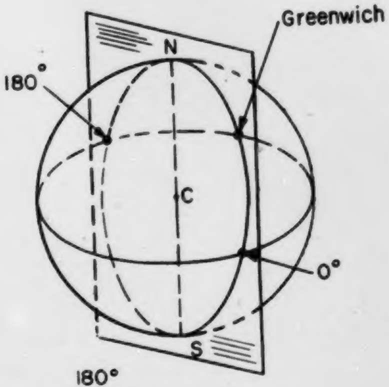


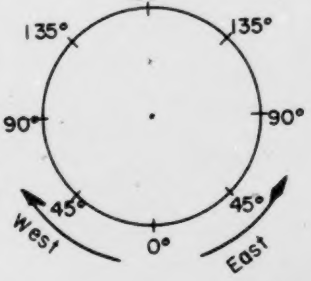
Let A be some point on the sphere as shown at the right. There is exactly one plane through A, perpendicular to the axis of the earth. This plane contains the parallel of latitude through A. There is exactly one meridian through A because the point A and the North Pole (or South Pole) determine one great circle. Since that great circle passes through the poles, the arc of the great circle containing A is a meridian. Thus, through each point of a sphere, except the poles, there is exactly one parallel of latitude and one meridian.

The zero meridian for the earth has been designated as the meridian which passes through a certain location in Greenwich (pronounced Gren - ich), England, near London. We sometimes refer to this meridian as the Greenwich meridian, even though the meridian itself passes through one particular point of the town. The meridian at Greenwich is sometimes called the prime meridian. This has nothing to do with a prime number.

The intersection of the Greenwich meridian and the equator is marked 0°. From this point, we follow the equator east, or west, until we reach the meridian which passes through the antipode of the Greenwich point, that is, lies on the great circle through Greenwich and the North Pole. This meridian intersects the equator at a point which is one-half way around the equator from the point labeled 0°. This point is labeled 180°. We can think of a plane intersecting the earth in this great circle. The plane separates the earth into two hemispheres, or half-spheres, as shown in the drawing at the right. The hemisphere on the left as you look at the drawing, is named the western hemisphere. The hemisphere on the right is the eastern hemisphere.

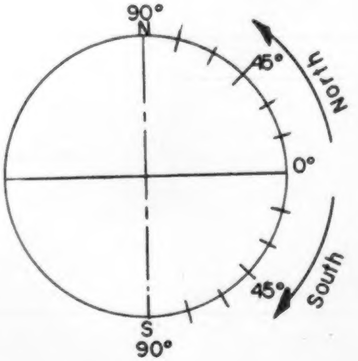
The great circle, which we call the equator, is divided into 360 equal parts in a view as seen from the North Pole. The whole numbers between 0 and 180 are assigned to the points on the half equator to the left of 0°. The same is done for the points on the other half of the equator. Each of these points names the meridian passing through that point. Any point on earth may be located by the meridian passing through the point. For example, Los Angeles is approximately on the meridian



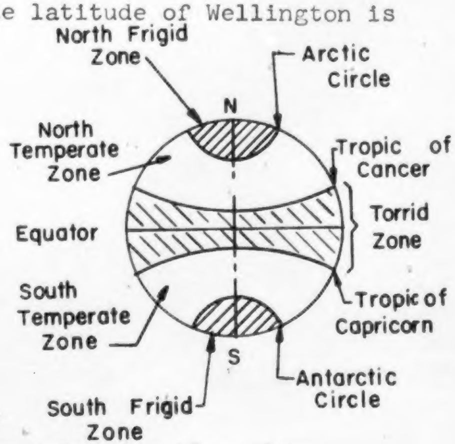


120° west of the Greenwich meridian. Tokyo is approximately on the meridian 140° east of Greenwich. We say the longitude of Los Angeles is about 120° W(west). The longitude of Tokyo is about 140° E.

The parallels of latitude are located in the following way. The equator is designated the zero parallel. All points above the equator are in the northern hemisphere, points below in the southern hemisphere. We choose any meridian, for instance that meridian through Greenwich. The part of the meridian from the intersection with the equator to the North Pole is divided into 90 equal parts, assuming that the earth is a sphere. The whole numbers between 0 and 90 are assigned to these points. Each point determines a parallel of latitude. A similar pattern is followed for points on the meridian south of the equator. For any point on earth, we may locate the parallel of latitude containing the point. For example, New Orleans is approximately on the parallel 30° north of the equator. Wellington, New Zealand is approximately on the parallel 40° south of the equator. We say that New Orleans has a latitude of about 30° N(north). The latitude of Wellington is approximately 40° S.



Some of the parallels are given special names. The Arctic and Antarctic Circles are the parallels located about $23\frac{1}{2}$ degrees from the North and South Poles. The Tropic of Cancer is about $23\frac{1}{2}$ degrees north of the equator, and the Tropic of Capricorn is the same distance south of the equator. Portions of spheres between two parallels of latitude are sometimes called zones. Some of these zones are also given special names as shown in the drawing at the right.



To locate a point on earth, we name the meridian and the parallel of latitude passing through the point. Thus we name the longitude and the latitude of a point. For example: 90° W, 30° N locates a point in the city of New Orleans. We say that New Orleans is located approximately at this point on earth. Durban, South Africa is located at approximately 30° E, 30° S. Note that the longitude is always listed first. Note that latitude and longitude give a coordinate system on the sphere just as the X-axis and Y-axis give a coordinate system in the plane.

Exercises 8-10

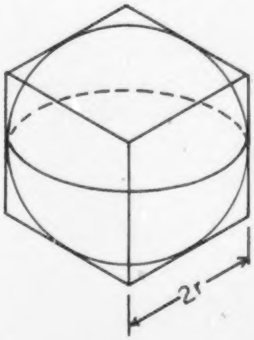
1. Using a globe, find the approximate location of each of the following cities. Indicate the location by listing the longitude first, followed by the latitude. Be sure to include the letters E or W and N or S in your answers.
   (a) New York City
   (b) Chicago
   (c) San Francisco
   (d) London
   (e) Paris
   (f) Moscow
   (g) Rio de Janeiro
   (h) Melbourne, Australia
2. Greenwich, England is located approximately on the parallel of latitude labeled 52° N. Without obtaining further information, write the location of Greenwich.
3. Chisimaio, Somalia, in eastern Africa, is located on the equator (or very near the equator). It is about 42 degrees east of Greenwich. Without using a reference, write the location of Chisimaio.
4. Using a map or a globe, find the cities located by the following:
   (a) 58° W, 35° S.
   (b) 175° E, 41° S.
5. Using a map or a globe, find the cities located by the following:
   (a) 122° E, 35° N.
   (b) 4° W, 41° N.
6. (a) Compare the location of the city in your answer for 4(a) with the location of the city in your answer for 5(a).
   (b) Similarly, compare the locations of the cities determined by 4(b) and 5(b).
   (c) What kind of points do these locations suggest?

7. Determine a way of finding the location of an antipodal point, say of 90° W, 45° N, without using a globe or map.
   Then find the antipodal points of each of the following:
   (a) 80° W, 25° S. (c) 180° W, 51° S.
   (b) 100° E, 65° N.
8. Where and what is the International Date Line?
9. Southeast of Australia, there is located a group of islands called the Antipodes Islands. They received this name because they are antipodal to Greenwich. Without using a reference, write the location of the Antipodes. When it is midnight in Greenwich, what time of day is it at the Antipodes? When it is the middle of summer in Greenwich, what season is it in the Antipodes?

8-11. Volume and Area of a Spherical Solid

A cube is a surface, and when we speak of the volume of a cube we mean the volume of a rectangular solid whose surface is a cube. Similarly, by volume of a sphere we mean the volume of a solid whose surface is a sphere.

The drawing at the right shows a sphere with the smallest possible cube built around the sphere. Each face of the cube is tangent to one point of the sphere. Hence the edge of the cube has measure $2r$ where r is the radius of the sphere.



$$\text{If } V_{cube} = e^3 \text{ (and } e = 2r\text{),}$$

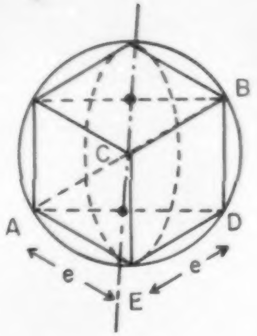
$$\text{then } V_{cube} = (2r)^3 \text{ or } 2\cdot r\cdot 2\cdot r\cdot 2\cdot r,$$

$$\text{or } V_{cube} = 8r^3$$

Since the sphere lies entirely within the cube,

$$V_{sphere} < 8r^3.$$

The drawing at the right represents a cube entirely within the sphere so that all the vertices of the cube lie on the sphere. Points A and B are opposite vertices of the cube. Point C is the center of the sphere and lies on the segment $\overline{AB}$. Hence A and B are antipodes.



Since ADE is a right triangle,

$$(AD)^2 = e^2 + e^2 \text{ or } 2e^2$$

(Note: AD means the measure of $\overline{AD}$).

Since ADB is also a right triangle,

$$(AD)^2 + (BD)^2 = (AB)^2$$

But, (BD) = e and (AB) = 2r

$$\text{so,} \quad 2e^2 + e^2 = (2r)^2$$

$$\text{and,} \quad 3e^2 = 4r^2.$$

Solving for e, $e = \sqrt{\frac{4}{3} r^2} = \frac{2r}{\sqrt{3}}$

Hence,

$$V_{cube} = e^3$$

$$V_c = \left(\frac{2r}{\sqrt{3}}\right)^3$$

$$V_c = \left(\frac{8\sqrt{3}}{9}\right) r^3 \approx 1.53r^3$$

Since the volume of the sphere is larger than the volume of this cube,

$$V_{sphere} > 1.53r^3.$$

$$\text{Therefore,} \quad 1.53r^3 < V_{sphere} < 8r^3$$

We have two numbers between which the measure of the volume of the sphere lies. We can see that these numbers are not very close to each other by replacing r with 2. Mathematicians have proved that the volume of a sphere may be found using the following formula,

$$V_s = \frac{4}{3} \pi r^3.$$

Note that $\frac{4}{3} \pi r^3$ lies between $1.53r^3$ and $8r^3$.

It is also difficult to determine a formula for the area of the surface of a sphere. Mathematicians have proved that the area of the surface of a sphere may be found by using the following formula:

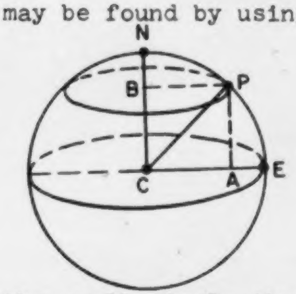
$$A_s = 4 \pi r^2.$$

Exercises 8-11

1. For each sphere whose radius is given below, find the volume of the corresponding spherical solid. Use 3.14 as the approximation for π.
   (a) $r = 3$ inches (c) $r = 4$ yards
   (b) $r = 10$ feet (d) $r = 6$ cm.
2. For the various spheres of Problem 1 find the surface area.
3. An oil tank is in the shape of a sphere whose diameter is 50 feet. The tank rests on a concrete slab.
   (a) If paint costs $6 per gallon and a gallon covers 400 square feet, find the cost of the paint for the surface.
   (b) If oil costs 13 cents per gallon find the value of the oil in the tank. Assume that the tank is full. (Use 1 cu. ft. = $7\frac{1}{2}$ gallons.)
4. A spherical balloon has a diameter of 40 feet. How much gas will it hold when all the air has been pushed out?
5. (a) If the radius of a sphere is doubled what effect does this have on the volume? On the surface area?
   (b) If the radius of a sphere is tripled what effect does this have on the volume? On the surface area?
6. Two spheres have radii in the ratio $\frac{3}{2}$.
   (a) Find the ratio of their volumes.
   (b) Find the ratio of their surface areas.

8-12. Finding Lengths of Small Circles

The lengths of circles of latitude may be found by using values of cosines of angles. In the drawing at the right, N represents the north pole, C the center of the earth, P some point on the surface of the earth, and E the point on the equator directly south of P. The drawing shows the great circles through E and through P, and the small circle through P.



The measure in degrees of $\angle PCE$ is the latitude of point P. Let A represent a point such that PAC is a right angle, and let B represent a point on $\overline{CN}$ such that PBC is a right angle. Thus, PBCA is a rectangle, and hence $\overline{PB}$ and $\overline{CA}$ are congruent.

Let e represent the length of the great circle represented by the equator. The radius of this circle is $\overline{CE}$, and $\overline{CE}$ and $\overline{CP}$ are congruent. The length of $e = 2\pi(CE) = 2\pi(CP)$.

Let p represent the length of the small circle at P. The radius of this circle is $\overline{BP}$, and $BP = CA$. Thus the length of $p = 2\pi(BP) = 2\pi(CA)$.

Let L° represent the latitude of P. Thus, the cosine of $PCA = \text{Cosine } L^{\circ}$,

$$\text{So, } \cos L^{\circ} = \frac{CA}{CP}$$

multiplying $\frac{CA}{CP}$ by 1, using $1 = \frac{2\pi}{2\pi}$, $\cos L^{\circ} = \frac{2\pi(CA)}{2\pi(CP)}$,

$$\text{Therefore, } \cos L^{\circ} = \frac{p}{e}.$$

Since $\cos L^{\circ} = \frac{p}{e}$, it follows that $p = e \cos L^{\circ}$. The length of the equator is about 25,000 miles, so $p \approx 25{,}000 \cos L^{\circ}$.

Assume the latitude represented by L° is 35°. The value for the cosine of 35° is 0.8192. The length of the small circle passing through a city having latitude of 35° is about

$$25{,}000 \times 0.8192 \approx 20{,}480 \text{ miles.}$$

Exercises 8-12

1. Find to the nearest ten miles the length of the circle of latitude which passes through the point with latitude given below.
   (a) 15° (b) 75° (c) 45°
2. How far is it between meridian 10° W. and 70° W. at latitude 40° N. along the parallel of latitude?
3. By sun-time is meant the time as determined by the position of the sun. Standard time zones should not enter into this problem.
   (a) If sun-time is 7:00 a.m. at meridian 10° W., find the sun-time at 70° W.
   (b) If sun-time is 7:00 a.m. at meridian 70° W., find the sun-time at 10° E.

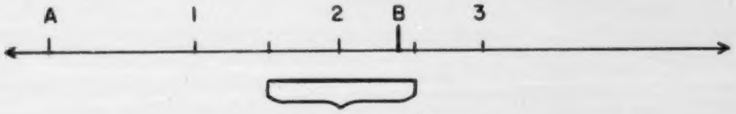
Chapter 9

RELATIVE ERROR

9-1. Greatest Possible Error

When you use numbers to count separate objects you need only counting numbers. When you measure something, the situation is different.

Scientists and mathematicians agree that measurement cannot be considered exact, but only approximate. The important thing to know is just how inexact a measurement may be, and to state clearly how inexact it may be.



The line above shows a scale divided into one-inch units. The zero point is labeled "A", and point B is between the 2-inch mark and the 3-inch mark. Since B is clearly closer to the two-inch mark, you may say to the nearest inch that the measurement of segment $\overline{AB}$ is 2 inches. However, any point which is more than $1\frac{1}{2}$ inches from A and less than $2\frac{1}{2}$ inches from A would be the endpoint of a segment whose length, to the nearest inch, is also 2 inches. The mark below the line shows the range within which the endpoint of a line segment 2 inches long (to the nearest inch) might fall. The length of such a segment might be almost $\frac{1}{2}$ inch less than 2 inches, or almost $\frac{1}{2}$ inch more than 2 inches. You therefore can say that, when a line segment is measured to the nearest inch, the greatest possible error is $\frac{1}{2}$ inch. Consequently such measurements are sometimes stated as $(2 \pm \frac{1}{2})$ inches. When a measurement is stated as 2 inches it may mean $(2 \pm \frac{1}{2})$ inches or (2 ± 0.5) inches. In the everyday world this is often not the case. However, in industrial and scientific work the greatest possible error should be specifically stated. For example, a measurement should be given as (2 ± 0.05) inches, or (2 ± 0.005) inches, not simply as 2 inches.

Exercises 9-1

1. Draw a line and mark on it a scale with divisions of $\frac{1}{4}$ inch. Mark the zero point C. Place a point between $1\frac{2}{4}$ and $1\frac{3}{4}$, but closer to $1\frac{3}{4}$, and call the point D. How long is $\overline{CD}$ to the nearest $\frac{1}{4}$ inch?
2. Between what two points on the scale must D lie if the measurement to the nearest $\frac{1}{4}$ inch, is to be $1\frac{3}{4}$ inches? How far from $1\frac{3}{4}$ inches is each of these points?
3. Why may this measurement of $\overline{CD}$ be stated $(1\frac{3}{4} \pm \frac{1}{8})$ inches?
4. The measure of a line segment was stated as $(2\frac{5}{16} \pm \frac{1}{32})$.
   (a) Between what marks on the scale must the end of this segment lie?
   (b) What is the greatest possible error?

9-2. Precision and Significant Digits

Consider the two measurements, $10\frac{1}{8}$ inches and $12\frac{1}{2}$ inches. As commonly used, these measurements do not indicate what unit of measurement was used. Suppose that the unit for the first measurement is $\frac{1}{8}$ inch, and the unit for the second measurement is $\frac{1}{2}$ inch. Then it is said that the first measurement is more precise than the second, or has greater precision. Notice also that the greatest possible error of the first measurement is $\frac{1}{2}$ of $\frac{1}{8}$ inch, or $\frac{1}{16}$ inch, and of the second is $\frac{1}{2}$ of $\frac{1}{2}$ inch, or $\frac{1}{4}$ inch. The greatest possible error is less for the first measurement than for the second measurement. Hence the more precise of two measurements is the one made with the smaller unit, and for which the greatest possible error is therefore the smaller.

In this chapter the convention, that the denominator of the fractional part of a measurement indicates the unit of measurement used, is adopted. If a line segment is measured to the nearest $\frac{1}{8}$ inch, and the measurement is $2\frac{6}{8}$ inches, you shall not change the fraction to $\frac{3}{4}$, for that would make it appear that the unit was $\frac{1}{4}$ inch, rather than $\frac{1}{8}$ inch. If a line segment is measured to the nearest $\frac{1}{4}$ inch, and the measurement is closer to 3 inches than to $2\frac{3}{4}$ or $3\frac{1}{4}$ inches, you shall state it to be $3\frac{0}{4}$ inches, so that it is clear that the unit used is $\frac{1}{4}$ inch.

Usually scientific measurements are expressed in decimal form. For instance, it is known that one meter is about 39.37 inches. This means that a meter is closer to 39.37 inches than it is to 39.38 inches or 39.36 inches. In other words, one meter lies between 39.375 inches and 39.365 inches.

The measurement, 39.37 inches includes 4 significant digits. They are significant in that they tell us the precision of our measurement. The place value of the last significant digit to the right indicates the precision, in this case one hundredth of an inch.

All non-zero digits are significant. A zero may or may not be significant. Zeros are significant when they are between non-zero digits as in numerals like 2007 (4 significant digits), 80,062 (5 significant digits), and 3.08 (3 significant digits). Zeros are not significant in numerals such as 0.008 and 0.026 because the zeros are used only to fix the decimal point.

If you are told that something is 73,000 feet long, it is not clear whether or not the zeros at the end are significant and actually indicate the precision. There is doubt about the precision of such a measurement. The unit of measurement may have been 1,000 feet, 100 feet, 10 feet, or 1 foot. In a case like this, a zero is sometimes underlined to show how precise the measurement is. For example, 73,$\underline{0}$00 feet (3 significant digits) means that the measurement is precise to the nearest 100 feet, 73,0$\underline{0}$0 feet (4 significant digits) means that the measurement is precise to the nearest 10 feet, and 73,00$\underline{0}$ feet (5 significant digits) means that the measurement is precise to the nearest foot. If no zero is underlined, you understand that the measurement was made to the nearest 1000 feet. If a measurement is stated as 5.640 feet you understand, without underlining the zero, that is is significant and that the unit is one thousandth of a foot, for otherwise the zero would not be written at all.

When a number is written in scientific notation, all of the digits in the first factor are significant. For example, the measurement 2.99776×10^{10} cm./sec., for the velocity of light, has 6 significant digits; the measurement 2.57×10^{-9} cm. for the radius of the hydrogen atom, has 3 significant digits; the measurement for the national debt in 1957, 2.8×10^{11} dollars, has 2 significant digits; 4.800×10^{8} has 4 significant digits. In the last case, the two final zeros are significant. Were they not, the number should have been written as 4.8×10^{8}.

Exercises 9-2

1. Suppose you measured a line to the nearest hundredth of an inch. Which of the following states the measurement best?

   3.2 inches   3.20 inches   3.200 inches

2. Suppose you measured to the nearest tenth of an inch. Which of the following should you use to state the result?

   4 inches   4.0 inches   4.00 inches

3. Tell which measurement in each pair has the greater precision.

   (a) 5.2 feet $2\frac{1}{4}$ feet
   (b) 0.68 feet,   23.5 feet
   (c) 0.235 inches,   0.146 inches.

4. (A) For each measurement below tell the place value of the last significant digit.

   (B) Tell the greatest possible error of the measurements.

   (a) 52700 feet   (c) 52.7 feet
   (b) $527\underline{0}$ feet   (d) 0.5270 feet

5. (a) Which of the measurements in Problem 4 is the most precise?
   (b) Which is the least precise?
   (c) Do any two measurements have the same precision?

6. Show by underlining a zero the precision of the following measurements:

   (a) 4200 feet measured to the nearest foot.
   (b) 48,000,000 people, reported to the nearest ten-thousand.

7. Tell the number of significant digits in each measurement:

   (a) 520 feet   (c) $25{,}8\underline{0}0$ ft.
   (b) 32.46 in.   (d) 0.0015 in.

8. How many significant digits are in each of the following:

   (a) 4.700×10^{5} (b) 6.70×10^{-4}

9-3. Relative Error, Accuracy and Percent of Error

While two measurements may be made with the same precision (that is, with the same unit) and therefore with the same greatest possible error, this error is more important in some cases than in others. An error of $\frac{1}{2}$ inch in measuring someone's height would not be very misleading, but an error of $\frac{1}{2}$ inch in measuring a nose would be misleading. One can get a measure of the importance of the greatest possible error by comparing it with the measurement. Consider these measurements and their greatest possible errors:

$$4 \text{ in.} \pm 0.5 \text{ in.}: \qquad 58 \text{ in.} \pm 0.5 \text{ in.}$$

Since these measurements are both made to the nearest inch, the greatest possible error in each case is 0.5 inch. If you divide the measure of the greatest possible error by the number of units in the measurement you will get these results: (Note that the measures are numbers and the measurements are not. The number of units in the measurement is called the measure).

$$\frac{0.5}{4} = \frac{5}{40} = 0.125$$

$$\frac{0.5}{58} = \frac{5}{580} \approx 0.0086$$

The quotients 0.125 and 0.0086 are called *relative errors*. The relative error of a measurement is defined as the quotient of the measure of the greatest possible error and the measure.

$$\text{Relative error} = \frac{\text{measure of the greatest possible error}}{\text{the measure}}$$

Percent of error is the relative error expressed as a percent. In the above two examples the relative error expressed as a percent is 12.5 and 0.86 . When written in this form it is called the percent of error.

The measurement with a relative error of 0.0086 (0.86) is more *accurate* than the measurement with a relative error of 0.125 (12.5). By definition a measurement with a smaller relative error is said to be more accurate than one with a larger relative error.

The terms *accuracy* and *precision* are used in industrial and scientific work in a special technical sense even though they are often used loosely and as synonyms in everyday conversation. Precision depends upon the size of the unit of measurement, which is twice the greatest possible error, while accuracy is the relative error or percent of error. For example, 12.5 pounds and 360.1 pounds are equally precise, that is, precise to the nearest 0.1 of a pound (greatest possible error in each case is 0.05 pound). The two measurements do not possess the same accuracy. The second measurement is more accurate. You should verify the last statement by computing the relative errors in each case and comparing them.

An astronomer, for example, making a measurement of the distance to a galaxy may have an error of a trillion miles (1,000,000,000,000 miles) yet be far more accurate than a machinist measuring the diameter of a steel pin to the nearest 0.001 inch.

Again, a measurement indicated as 3.5 inches and another as 3.5 feet are equally accurate but the first measurement is more precise. Why?

Exercises 9-3

In all computation express your answer so that it includes two significant digits.

1. State the greatest possible error for each of these measurements.
   (a) 52 ft. (c) 7.03 in.
   (b) 4.1 in. (d) $54{,}0\underline{0}0$ mi.
2. Find the relative error of each measurement in Problem 1.
3. Find the greatest possible error and the percent of error for each of the following measurements.
   (a) 9.3 ft. (b) 0.093 ft.
4. What do you observe about your answers for Problem 3? Can you explain why the percents of error should be the same for these measurements?
5. Find the precision of the following measurements.
   (a) 26.3 ft. (b) 51,000 mi.
6. How many significant digits are there in each of the following measurements?
   (a) 52.1 in. (c) 3.68 in.
   (b) 52.10 in. (d) 368.0 in.

7. Find the relative error of each of the measurements in Problem 6.
8. From your answers for Problems 6 and 7, can you see any relation between the number of significant digits in a measurement and its relative error? What is the relation between the number of significant digits in a measurement and its accuracy?
9. Without computing, can you tell which of the measurements below has the greatest accuracy? Which is the least accurate?
   23.6 in.   0.043 in.   7812 in.   0.2 in.
10. Arrange the following measurements in the order of their precision (from least to greatest).
   (a) $36\frac{1}{2}$ in., $27\frac{0}{16}$ in., $32\frac{3}{8}$ in., $46\frac{2}{7}$ in., $22\frac{1}{4}$ in.
   (b) 4.62 in., 3.041 in., 3 in., 82.4 in., 0.3762 in.
11. Count the number of significant digits in each of the following:
   (a) 43.26
   (b) 4,607
   (c) 0.6070
   (d) 0.0030
   (e) $76{,}00\underline{0}$
   (f) 43,000

9-4. Operations with Measures

Since measurements are never exact, the answers to any questions which depend on those measurements are also approximate. For instance, suppose you measured the length of a room by making two marks on a wall, which you called A and B, and then measuring the distances from the corner to A, from A to B, and from B to the other corner. Measurements such as these whose measures are to be added, should all be made with the same precision. Suppose, to the nearest fourth of an inch, the measurements were $72\frac{1}{4}$ inches, $40\frac{2}{4}$ inches, $22\frac{3}{4}$ inches. You would add the measures to get $135\frac{2}{4}$. Therefore the measurement is $135\frac{2}{4}$ inches. Of course, the distances might have been shorter in each case. The measures could have been almost as small as $72\frac{1}{8}$, $40\frac{3}{8}$, and $22\frac{5}{8}$ in which case the distance would have been almost as small as $135\frac{1}{8}$ inches, which is three-eighths of an inch less than $135\frac{2}{4}$ inches. Also, each distance might have been longer by nearly one-eighth of an inch, in which case the total length might have been almost three-eighths of an inch longer than $135\frac{2}{4}$ inches. The greatest possible error of a sum is the sum of the greatest possible errors. If you were adding measures of 37.6, 3.5, and 178.6, the greatest possible error of the sum would be 0.5 + 0.5 + 0.5 or 1.5 . The result of this addition could be shown as 219.7 ± 1.5 .

Computation involving measures is very important in today's world. Many rules have been laid down giving the accuracy or precision of the results obtained from computation with approximate measures.

The general principle is that the sum or difference of measures cannot be more precise than the least precise measure involved. Therefore to add or subtract numbers arising from approximations, first round each number to the unit of the least precise number and then perform the operation.

When data are expressed in decimal form a rough guide can be employed for finding a satisfactory product. *The number of significant digits in the product of two numbers is not more than the number of significant digits in the less accurate factor.* For example: The area of the rectangle 10.4 cm. by 4.7 cm. would be stated as 49 sq. cm.

$$\begin{array}{r} 10.4 \\ \underline{4.7} \\ 728 \\ \underline{416} \\ 48.88 \end{array}$$

If one of the two factors contains more significant digits than the other, round off the factor which has more significant digits so that it contains only one more significant digit than the other factor.

Exercises 9-4

1. Find the greatest possible error for the sums of the measurements for each of the following:
   (a) $5\frac{1}{2}$ in., $6\frac{1}{2}$ in., $3\frac{0}{2}$ in.
   (b) 0.004 in., 2.1 in., 6.135 in.
   (c) $2\frac{3}{4}$ in., $1\frac{5}{16}$ in., $3\frac{3}{8}$ in.
2. Add the following measures:
   (a) 42.36, 578.1, 73.4, 37.285, 0.62
   (b) 85.42, 7.301, 16.015, 36.4
3. Subtract the following measures:
   (a) 7.3 - 6.28 (b) 735 - 0.73 (c) 5430 - 647

4. Multiply the following approximate numbers.
   (a) 4.1×36.9 (b) 3.6×4673 (c) $3.76 \times (2.9 \times 10^4)$
5. Find the area of a rectangular field which is 835.5 rods long and 305 rods wide.
6. A machine stamps out parts each weighing 0.625 lb. How much weight is there in 75 of these parts?
7. Assuming that water weighs 62.5 lb. per cu. ft., what is the volume of 15,610 lbs.?

There are many rough rules for computing with approximate data but they have to be used with a great deal of common sense. They don't work in all cases. The modern high speed computing machine which adds or multiplies thousands of numbers per second has to have special rules applied to the data which are fed to it. Errors involved in rounding numbers add up or disappear in a very unpredictable fashion in these machines. As a matter of fact "error theory" as applied to computers is an active field of research today for mathematicians.